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From Calculus to Analysis

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To Tina, with love

Preface

I have taught elementary analysis many times. It is a difficult course for students and instructor alike. Students lack basic techniques such as manipulations of simple inequalities with elementary functions. For instance, most students will have trouble finding upper and lower numerical bounds for $\frac{1}{1+x^2}$ when $-4 < x < -2$; they also lack mathematical culture at this stage. Many students have no idea why we need limits, why series are important, what the number π is, and so on. These gaps are actually not that surprising given how little theory students have been exposed to up to that point.

To decrease the failure rate in analysis and make it a less traumatic experience for both the students and the instructor, we have offered a pre-analysis course for the last several years at the University of Colorado at Colorado Springs. Students are strongly advised to take one semester of pre-analysis, and they then take the required one semester of analysis. The experiment has been remarkably successful. The failure rate in analysis has dropped significantly, and I get many more positive comments about the whole learning experience. The only problem is that there are rather few textbooks available for a pre-analysis course. The main goal of this work is to provide such a textbook.

I use Chaps. 1 through 4 for my pre-analysis course. My goal is to get students comfortable at estimating simple algebraic expressions and at the same time increase their mathematical culture. In order to achieve these two goals simultaneously, the order in which topics appear is not the traditional one, and many concepts are introduced several times. For instance, I compute derivatives in Chap. 3 long before differentiation is defined in Chap. 5.

I have also used my notes that evolved into this book for a classical elementary analysis course (1.3, 2.1, 2.2, Chaps. 5 and 6, and selected topics from Chaps. 7, 8, and 9). Chapter 7 is a short introduction to uniform convergence. It also contains the proofs of the classical results on differentiation and integration of power series that are used in Chaps. 3 and 4. Chapters 8 and 9 are introductions to decimal representations of the reals and countability, respectively. Chapter 8 is the closest we get to constructing the reals, and Chap. 9 contains important ideas and results that students need to be exposed to at this point in their mathematical education and career. The chapters of the book are largely independent from each other, and the only prerequisite is a standard calculus course.

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Contents

Chapter 1
Number Systems

1.1 The Algebra of the Reals

We start with some notation and symbols. We denote the naturals by $\mathbf{N} = \{1, 2, 3, \ldots\}$, the integers by $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the rationals (represented by fractions of integers) by $\mathbf{Q}$, and the reals (which may be described by their decimal representation, see Chap. 8) by $\mathbf{R}$. The symbol $\in$ means 'belongs to'. For instance, $1 \in \mathbf{N}$ means '1 is a natural'. The notation

$$\left\{x \in \mathbf{R} : x^2 = 2\right\}$$

designates the set of reals whose square is 2. The set

$$\{x \in \mathbf{R} : -1 < x \leq 2\}$$

is the set of real numbers strictly larger than -1 and smaller than or equal to 2. A shorter notation for such a set is $(-1, 2]$. The set

$$\left\{x^2 : x \in [-1, 1]\right\}$$

is the set of squares of numbers in $[-1, 1]$. The notation

$$\{-1, 2, 4\}$$

designates a set with three elements: -1, 2, and 4. The empty set will be denoted by $\emptyset$.

At the beginning of any mathematics (geometry, probability, ...) there is a set of rules called axioms. These rules are assumed and not proved. Everything else is proved using these axioms. For instance, all of Euclidean geometry is based on 5 axioms.

The Peano axioms may be used to define the set of naturals $\mathbf{N}$, the addition operation, and the order relation $<$. Then one can construct the integers, the rationals, and finally the reals. However, this is a rather involved program and is outside the scope of this text. We will not construct the number systems. The reader may consult Goodfriend (2005) for a discussion of the Peano axioms and their consequences and Krantz (1991) for the construction of integers, rationals, and reals, see the references at the end of the book.

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We will not recall the familiar properties of the reals regarding addition, multiplication, distributivity, and so on. We will however recall the familiar operations regarding inequalities. First, we give a set of rules.

Ordering the reals
The binary relation $<$ has the following properties.

R1. For x and y in $\mathbf{R}$, there are three possibilities: $x = y$, $x < y$, or $y < x$.
R2. If $x < y$ and $y < z$, then $x < z$.
R3. If $x < y$ and z is any real, then $x + z < y + z$.
R4. If $0 < x$ and $0 < y$, then $0 < xy$.

We may use $y > x$ for $x < y$. The notation $x \leq y$ means that $x < y$ or $x = y$. We now prove some useful consequences.

C1. If $x < y$ and $z > 0$, then $xz < yz$.

By R3 we have $x - x < y - x$. Hence, $0 < y - x$. By R4 we have

$$(y - x)z > 0.$$

That is,

$$yz - xz > 0.$$

Thus, by R3

$$yz > xz.$$

This proves C1.

C2. If $x < y$ and $z < 0$, then $xz > yz$.

The proof is very similar to the proof of C1 and is left as an exercise.

C3. If $0 < x < y$, then $x^2 < y^2$.

Since $x < y$ and $x > 0$, by C1 $xx < xy$. Similarly, since $x < y$ and $y > 0$, by C1 $xy < yy$. By R2, $x^2 < y^2$.

C4. If $x > 0$, then $1/x > 0$.

Let $x' = 1/x$. If we had $x' < 0$, then by C1 we would have $x'x < 0x = 0$. But $x'x = 1 > 0$. Thus, $x' > 0$.

C5. If $x > y > 0$, then $1/x < 1/y$.

By R4 $xy > 0$, and so by C4 $\frac{1}{xy} > 0$. By C1,

$$\frac{1}{xy}x > \frac{1}{xy}y.$$

That is, $1/y > 1/x$.

We now turn to the function *absolute value*. We define

$$|x| = x \quad \text{if } x \geq 0,$$
$$|x| = -x \quad \text{if } x \leq 0.$$

The following simple lemma is very useful.

Lemma *Let $a > 0$ be a real number. We have $|x| \leq a$ if and only if $-a \leq x \leq a$.*

We prove the lemma. It is an 'If and only if' statement, which means that we have two implications to prove.

For the direct implication, assume that $|x| \leq a$. There are two cases. If $x \geq 0$, $|x| = x$ and $x \leq a$. Since $x \geq 0$, we must have $-a \leq x$. Thus, $-a \leq x \leq a$. On the other hand, if $x \leq 0$, $|x| = -x$ and $-x \leq a$, that is, $x \geq -a$. Since $x \leq 0$, we have $x \leq a$. In both cases $-a \leq x \leq a$, and the direct implication is proved.

For the converse, assume now that $-a \leq x \leq a$. If $|x| = x$, then $|x| \leq a$. If $|x| = -x$, since $-a \leq x$, we have $a \geq -x$ and $a \geq |x|$. In both cases $|x| \leq a$, and the lemma is proved.

Triangle inequality
For any real numbers a and b, we have

$$|a+b| \leq |a| + |b|.$$

For any reals a and b, we have

$$-|a| \leq a \leq |a| \quad \text{(why?)},$$
$$-|b| \leq b \leq |b|.$$

By adding the inequalities we get

$$-\big(|a| + |b|\big) \leq a + b \leq |a| + |b|.$$

According to the lemma above, this implies that

$$|a+b| \leq |a| + |b|.$$

This completes the proof of the triangle inequality.

We now turn to an important algebraic identity. We define the nth power of a real a by

$$a^1 = a$$

and

$$a^{n+1} = a^n a \quad \text{for all } n \geq 1.$$

We also set $a^0 = 1$.

An algebraic identity
For any natural number n and any real numbers a and b, we have that

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k.$$

For $n = 2$ and $n = 3$, these are the well-known identities

$$a^2 - b^2 = (a-b)\sum_{k=0}^{1} a^{1-k}b^k = (a-b)(a+b)$$

and

$$a^3 - b^3 = (a-b)\sum_{k=0}^{2} a^{2-k}b^k = (a-b)\big(a^2 + ab + b^2\big).$$

For $n \geq 4$, the identity is

$$a^n - b^n = (a-b)\big(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\big).$$

We now prove it. We have

$$(a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k = a\sum_{k=0}^{n-1} a^{n-1-k}b^k - b\sum_{k=0}^{n-1} a^{n-1-k}b^k.$$

Thus,

$$(a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k = \sum_{k=0}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-1} a^{n-1-k}b^{k+1}.$$

The difference between the two sums is

$$a^n + a^{n-1}b + \cdots + a^2b^{n-2} + ab^{n-1} - \big(a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n\big).$$

All the terms cancel except $a^n - b^n$. This proves the identity above.

Exercises

1. (a) Describe in words the set

$$\big\{x \in \mathbf{Q} : x^2 < 3\big\}.$$

   (b) Describe with mathematical symbols the set of reals whose inverses are between 1 and 2.
   (c) Describe in words the set $(-1, 5)$.
2. Let $A = \{\frac{1}{n} : n \in \mathbf{N}\}$.
   (a) Pick an element in A.
   (b) Describe in words the set A.

3. Prove that if $x < y$ and $z < 0$, then $xz > yz$.
4. (a) Assume that $x < y < 0$. State an inequality between x^2 and y^2 and prove it.
   (b) Assume that $x < y < 0$. State an inequality between $1/x$ and $1/y$ and prove it.
5. (a) Show that if $0 < a < 1$ and $x \geq 0$, then $ax \leq x$.
   (b) Show that if $0 < a < 1$ and $x \geq 0$, then $x/a \geq x$.
   (c) Show that if $x \geq 0$, then
$$\frac{x}{x+1} \leq x.$$
6. Assume that $a > 1$ and $x < 0$.
   (a) Compare ax and x.
   (b) Compare x/a and x.
7. Prove that for any real number a, we have $-|a| \leq a \leq |a|$.
8. When is the triangle inequality $|a+b| \leq |a| + |b|$ an equality?
9. (a) Show that
$$|a| - |b| \leq |a-b|.$$
   (b) Show that
$$|b| - |a| \leq |b-a| = |a-b|.$$
   (c) Show that
$$\big||a| - |b|\big| \leq |a-b|.$$
10. Assume that $|a-b| < \epsilon$. Show that
    (a) $b - \epsilon < a < b + \epsilon$.
    (b) $|a| < |b| + \epsilon$.
    (c) $|a| > |b| - \epsilon$.
11. Suppose that $a \leq x \leq b$ and $a \leq y \leq b$. Show that
$$|x-y| \leq b-a.$$
12. Assume that $-1 \leq x \leq 2$ and $3 \leq y \leq 4$. Find a lower and an upper bound for x/y.
13. Assume that $0 \leq x < 1$.
    (a) Show that
$$\frac{1}{1-x} \geq 1 + x + x^2.$$
    (b) Generalize (a).
14. Show that if $a \neq 1$ and n is a natural, then
$$1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}.$$
15. Let $a > 0$ be a real number. Find d such that the set $(a-d, a+d)$ has only strictly positive numbers.

16. (a) Show that for $1/4 \le x \le 3/4$, we have
$$\frac{x}{1-x} \ge \frac{4}{3}x.$$
(b) Find a constant c so that for $1/4 \le x \le 3/4$, we have
$$\frac{x}{1-x} \le cx.$$

1.2 Natural Numbers and Integers

We first need the notion of minimum.

Minimum of a set
A natural number n_0 is said to be the minimum of a nonempty subset A of the naturals if the following two conditions hold:

(i) n_0 belongs to A.
(ii) If n belongs to A, then $n \ge n_0$.

We denote the minimum of A (if it exists) by $\min A$. We now state three easy consequences of this definition.

C1. If a minimum exists, it is unique. Thus, we may talk about *the* minimum of A.

Assume that A has two minima n_0 and n_1. Since n_0 is a minimum and n_1 is in A, we have by definition that $n_1 \ge n_0$. By exchanging n_0 and n_1 in the preceding sentence we get $n_0 \ge n_1$. Thus, $n_0 = n_1$. That is, there is only one minimum.

C2. If A is a singleton, then it has a minimum, and that minimum is the only element in A.

Assume that $A = \{p\}$. Then p is in A, and if n is in A, then n is necessarily p, and therefore $n \ge p$. That is, p satisfies the two requirements of the definition above, and we get $\min A = p$.

C3. If $A \subset B$ and they both have a minimum, then $\min A \ge \min B$.

This is so because $\min A \in A \subset B$, and so $\min A$ is in B. But $\min B$ is the smallest of the elements in B. Thus, $\min A \ge \min B$.

With the five Peano's axioms one can construct the naturals, define the addition, and the order relation $<$. Here we only state one of the axioms: the well-ordering principle.

Well-ordering principle
If A is a nonempty subset of naturals, then it has a minimum.

Example 1.1 Let r be a rational such that $0 < r < 1$. Show that r can be written in irreducible form. That is, r may be written as the ratio of two naturals that have no common divisors but 1.

Let

$$A = \{n \in \mathbf{N} : nr \in \mathbf{N}\}.$$

Since r is a rational, there are naturals a and b such that

$$r = \frac{a}{b}.$$

Therefore, $rb = a$, and since a is a natural, b is in A. Hence, A is a nonempty subset of naturals. By the well-ordering principle, A has a minimum m.

Let $k = mr$. Since m is in A, k is a natural. We now need to show that k and m are relatively prime. We do a proof by contradiction. Assume that $d > 1$ divides both k and m. Therefore, there are naturals k' and m' such that

$$k = k'd \quad \text{and} \quad m = m'd.$$

Since $k = mr$, we get $k'd = m'dr$. Thus, $k' = m'r$. In particular, m' must be in A, but $m' < m$, the minimum of A. Hence, we have a contradiction: the naturals k and m have no common divisor, and $r = k/m$ is irreducible.

Principle of induction
Let P be a subset of $\mathbf{N}$. Assume the following two properties for P:

(i) 1 belongs to P.
(ii) If n belongs to P, then $n + 1$ belongs to P.

Then P is all of $\mathbf{N}$.

We now prove the principle of induction. Let T be the set of all naturals that are not in P. We want to show that T is empty, so that $P = \mathbf{N}$. We do a proof by contradiction. That is, we assume that T is nonempty, and we use our axioms to show that this leads to a statement that is not true. This forces to conclude that T must be empty.

If T is nonempty, by the well-ordering principle, T has a minimum that we denote by a. Since 1 belongs to P, a cannot be 1. We must have $a > 1$. The number $a - 1$ is a natural (since $a > 1$) and cannot be in T since a is the minimum of T. Thus, $a - 1$ is in P. By (ii), $a - 1 + 1 = a$ is also in P. That is, a is in P. But a is also in T, that is, not in P. We have a contradiction: a is in P and is not in P. Hence, T is empty, and the principle of induction is proved.

We will now see an application of the induction principle.

Example 1.2 Show that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Let P be the set of naturals whose sum of the first n integers is $\frac{n(n+1)}{2}$. More mathematically, let

$$S_n = \sum_{i=1}^{n} i \quad \text{and} \quad a_n = \frac{n(n+1)}{2}.$$

Then

$$P = \{n \in \mathbf{N} : S_n = a_n\}.$$

That is, P is the set of naturals n for which $S_n = a_n$. We have $S_1 = 1$ and $a_1 = 1$, so 1 belongs to P.

Now assume that n belongs to P so that $S_n = a_n$. Consider

$$S_{n+1} = 1 + 2 + \cdots + n + (n+1) = S_n + (n+1) = a_n + n + 1,$$

where the last equality follows from the assumption that n is in P. Since

$$a_n + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2} = a_{n+1},$$

we have

$$S_{n+1} = a_{n+1}.$$

This proves that $n+1$ belongs to P. Thus, by the principle of induction, $P = \mathbf{N}$. Therefore, for every natural number n, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Example 1.3 Define factorials by

$$n! = 1 \times 2 \times \cdots \times n.$$

Prove that for every natural n,

$$n! \geq 2^{n-1}.$$

Let

$$P = \left\{n \in \mathbf{N} : n! \geq 2^{n-1}\right\}.$$

For $n = 1$, we have $n! = 1! = 1$ and $2^{n-1} = 2^0 = 1$. Hence, 1 is in P. Assume now that n is in P. Then

$$(n+1)! = n!(n+1) \geq 2^{n-1}(n+1).$$

Since $n \geq 1$, we have $n + 1 \geq 2$ and

$$(n+1)! \geq 2^{n-1}(n+1) \geq 2^{n-1}2 = 2^n = 2^{n+1-1}.$$

That is, $n+1$ belongs to P.

We now state an alternative form of the induction principle that is useful.

Alternative form of the principle of induction
Let $S(n)$ be a statement for natural n. Assume that

(i) $S(1)$ is true;
(ii) If $S(n)$ is true, then $S(n+1)$ is also true.

Then $S(n)$ is true for all n in $\mathbf{N}$.

To prove this form of the induction principle, we define

$$P = \{n \in \mathbf{N} : S(n) \text{ is true}\}.$$

That is, P is the set of naturals for which $S(n)$ is true. Since $S(1)$ is true, we have 1 in P. Assume that n is in P. By definition of P, $S(n)$ is true. By property (ii), we have that $S(n+1)$ is true, and so $n+1$ is in P. By the principle of induction, all naturals are in P. That is, $S(n)$ is true for all naturals n. We have proved the alternative form of the principle of induction.

We use the alternative form of the principle of induction to prove the following inequality.

Bernoulli's inequality
For any real $a \geq 0$ and any natural n, we have

$$(1+a)^n \geq 1 + na.$$

We now prove this inequality. Define the sequences

$$a_n = (1+a)^n \quad \text{and} \quad b_n = 1 + na.$$

For $n \geq 1$, let $S(n)$ be the statement: $a_n \geq b_n$. Take $n = 1$. Then

$$a_1 = (1+a)^1 = 1 + a = b_1.$$

That is, $S(1)$ is true. Assume now that $S(n)$ is true. Since $a_n \geq b_n$, we have

$$a_n(1+a) \geq b_n(1+a),$$

and therefore,

$$a_{n+1} = (1+a)^{n+1} = (1+a)^n(1+a) = a_n(1+a) \geq b_n(1+a).$$

By the definition of b_n we have

$$\begin{aligned} b_n(1+a) &= (1+na)(1+a) = 1 + a + na + na^2 \\ &\geq 1 + a + na = 1 + (n+1)a = b_{n+1}. \end{aligned}$$

Since $a_{n+1} \geq b_n(1+a)$, we have

$$a_{n+1} \geq b_{n+1}.$$

That is, $S(n+1)$ is true. By the alternative form of the principle of induction, $S(n)$ is true for every $n \geq 1$. This completes the proof of Bernoulli's inequality.

Once the natural numbers are given, it is not difficult to construct the number 0 and the negative integers. The interested reader may refer, for instance, to Krantz (1991) (see references below). We denote by $\mathbf{Z}$ the set of all integers (negative, positive, and 0).

We now introduce long division.

Long division in the naturals
Given two natural numbers a and b, there exist positive integers q and r such that

$$a = bq + r,$$

where $0 \leq r < b$. Moreover, q and r are unique. If $r = 0$, a is said to be divisible by b.

We start by showing that q and r are unique. Assume that $a = bq + r$ and that $a = bq' + r'$. Assume also that $0 \leq r < b$ and $0 \leq r' < b$. Subtracting the two equalities, we get

$$0 = b(q - q') + (r - r').$$

In particular,

$$r' - r = b(q - q').$$

Since

$$0 \leq r' < b$$

and

$$-b < -r \leq 0,$$

we get by addition $-b < r' - r < b$. But if $q \neq q'$, then either $q - q' \geq 1$ and hence $b(q - q') \geq b$ or $q - q' \leq -1$ and hence $b(q - q') \leq -b$. In both cases, $b(q - q')$ cannot be equal to $r' - r$ (since $r' - r$ is strictly between $-b$ and b). Thus, $q = q'$ and $r = r'$. This shows the uniqueness of q and r.

Now we deal with the existence of q and r. Consider the set

$$A = \{n \in \mathbf{N} : bn > a\}.$$

Note that, since $b \geq 1$, $b(a+1) \geq a+1 > a$. Thus, $a+1$ is in A, and $A \neq \emptyset$. By the well-ordering principle, A has a minimum. Let $\min A = n_0$. There are two possibilities. If $n_0 = 1$, then 1 is in A, and so $b > a$. We can set $q = 0$, $r = a$, and we have

$$a = qb + r$$

with $0 \le r < b$.

On the other hand, if $n_0 > 1$, then we set $q = n_0 - 1$ and $r = a - bq$. Note that $n_0 - 1$ is not in A since n_0 is the minimum of A. There are two ways for $n_0 - 1$ not to be in A. Either it is not a natural, or $b(n_0 - 1) \le a$. Since $n_0 > 1$, $n_0 - 1$ is a natural. Thus, $bq = b(n_0 - 1) \le a$, so that $r = a - bq \ge 0$. Since n_0 is in A, we have $a - bn_0 < 0$, and therefore,

$$r = a - bq = a - b(n_0 - 1) = a - bn_0 + b < b.$$

We have proved the existence of r and q.

Example 1.4 Show that every natural number n can be written as $n = 2m + 1$ or as $n = 2m$. In the first case the number is said to be odd, and in the second it is said to be even.

We do the long division of n by 2. There are positive integers m and r such that

$$n = 2m + r \quad \text{where } 0 \le r < 2.$$

Hence, $n = 2m$ or $2m + 1$. Since r is unique, a natural may be odd or even but not both.

Example 1.5 Show that for any natural n, the two naturals n and n^2 are either both even or both odd.

If n is even, then $n = 2m$ for some positive integer m. Thus, $n^2 = 4m^2 = 2(2m^2) = 2k$, where $k = 2m^2$ is a positive integer. Hence, n^2 is even.

If n is odd, then $n = 2a + 1$ for some positive integer a. Thus,

$$n^2 = 4a^2 + 4a + 1 = 2\big(2a^2 + 1\big) + 1 = 2a' + 1,$$

where $a' = 2a^2 + 1$ is a positive integer. Therefore, n^2 is odd.

Exercises

1. Prove that for any n in $\mathbf{N}$, we have

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. (a) Find a formula for

$$1^3 + 2^3 + \cdots + n^3.$$

   (b) Prove the formula in (a).
3. Prove that for any natural number n, we have

$$2^n > n.$$

4. Let E_1, $E_2, \dots, E_n$ be subsets of a set X. Assume that x is in at least one of the E_i. Define the set

$$I = \{i \in \mathbf{N} : i \leq n \text{ and } x \in E_i\}.$$

   (a) Show that I has a minimum that we denote by $m(x)$.
   (b) Show that if $m(x) > 1$, then

$$x \in E_{m(x)} \cap \left(\bigcup_{i=1}^{m(x)-1} E_i \right)^c,$$

   where A^c denotes the complement of a set A.
5. Find and prove a formula for the sum of the first n even naturals.
6. Find and prove a formula for the sum of the first n odd naturals.
7. (a) Give a definition for the maximum of a set integers.
   (b) If A is nonempty, does $\max A$ necessarily exist?
   (c) Assume that A and B are two sets of integers such that $A \subset B$. Assume that $\max A$ and $\max B$ exist. Find a relation between these two numbers.
8. Recall that a prime number is a natural number larger than or equal to 2 and such that its only divisors are 1 and itself. Let $n \geq 2$ be a natural. Let

$$A = \{k \in \mathbf{N} : k \geq 2 \text{ and } k \text{ divides } n\}.$$

   (a) Show that A has a minimum m.
   (b) Prove that m is prime.
   (c) Show that n is divisible by a prime.
9. In this exercise we prove that there are infinitely many prime numbers using Euclid's argument.
   (a) Let p be a prime number, and let

$$q = 2 \times 3 \times 5 \times \cdots \times p + 1.$$

   That is, q is the product of all prime numbers up to p plus 1. Show that no prime number less than or equal to p divides q.
   (b) Show that there must be a prime number strictly larger than p (use Exercise 8).
   (c) Conclude that there are infinitely many prime numbers.
10. A prime number is a natural number (strictly larger than 1) which is only divisible by 1 and itself.
   (a) Show that if $a \geq 2$ is a natural number such that $a^n - 1$ is prime for some natural number $n \geq 2$, then $a = 2$ (factor $a^n - 1$).
   (b) Show that for natural numbers s and t, we have

$$2^{st} - 1 = \left(2^s - 1\right)\left(2^{s(t-1)} + 2^{s(t-2)} + \cdots + 1\right).$$

   (c) Show that if $2^n - 1$ is a prime number, then n is also a prime number ($2^n - 1$ is then called a Mersenne number).
   (d) State the converse of the property stated in (c).
   (e) Is the converse stated in (d) true? (Note that $2^{11} - 1 = 2047 = 23 \times 89$).

11. (a) Show that for any natural numbers k and n, we have

$$\sum_{j=1}^{n}\left((j+1)^k - j^k\right) = (n+1)^k - 1.$$

(b) Show that

$$\sum_{j=1}^{n}\left((j+1)^2 - j^2\right) = \sum_{j=1}^{n}(2j+1).$$

(c) Let $S_k(n) = \sum_{j=1}^{n} j^k$. Use (a) and (b) to show that

$$(n+1)^2 - 1 = 2S_1(n) + n$$

and solve for $S_1(n)$.

(d) To find $S_2(n)$, do steps (b) and (c) starting by expanding $\sum_{j=1}^{n}((j+1)^3 - j^3)$.

(e) Find $S_3(n)$ and $S_4(n)$.

1.3 Rational Numbers and Real Numbers

Natural numbers are good for counting, but as soon as we want to measure things, we need fractions. If we use the meter as a unit of length, then the object we are measuring will rarely be exactly a whole number of meters, and we will need fractions of meters to have a more precise measurement. This is why we need $\mathbf{Q}$ the set of rational numbers. We will not formally construct $\mathbf{Q}$. We will think of a rational number as being represented by a fraction of integers. The same rational may be represented by infinitely many fractions ($1/2 = 2/4 = 3/6 = \cdots$).

Interestingly, the rationals are not enough to measure lengths. Consider a right isosceles triangle with sides 1, 1, and x. Then, by Pythagoras we have

$$x^2 = 1^2 + 1^2.$$

So x is such that $x^2 = 2$. There is at most one positive solution to this equation (why?), and we denote it (assuming that it exists!) by $\sqrt{2}$. It turns out that $\sqrt{2}$ cannot be a rational number. The ancient Greeks already knew that. There is actually nothing special about $\sqrt{2}$: $\sqrt{n}$ is either a natural number (when n is a perfect square) or an *irrational* (when n is not a perfect square), see the exercises.

We now show that $\sqrt{2}$ is not a rational. We do a proof by contradiction. That is, we assume that $\sqrt{2}$ is rational, and we show that this leads to a contradiction.

Assume that there are natural numbers a and b such that $\sqrt{2} = a/b$ and a/b is irreducible (see Example 1.1 in Sect. 1.2). In particular, a and b are not both even. By definition, $\sqrt{2}$ is a solution of the equation $x^2 = 2$. Hence,

$$\sqrt{2}^2 = 2.$$

So

$$a^2/b^2 = 2.$$

That is, $a^2 = 2b^2$, and a^2 is even. But a^2 is even if and only if a is even (see Example 1.5 in Sect. 1.2). Thus, there is a natural number a' such that $a = 2a'$, and so $a^2 = 4a'^2 = 2b^2$. We get $b^2 = 2a'^2$, and therefore b^2 and b are even. But we assumed that a and b are not both even. This provides a contradiction: if we assume that $\sqrt{2}$ is rational, we end up with something absurd 'a and b are not both even, and yet they are both even'. This shows that our starting assumption '$\sqrt{2}$ is rational' cannot be true.

Thus, we need yet another set of numbers, the *real numbers*. Building the real numbers from the rationals is rather subtle. To obtain the real numbers from the rationals, an algebraic construction, such as the ones to go from the natural numbers to the integers and from the integers to the rationals, is not enough. The real numbers are *limits* of rational sequences. We will not construct the reals, but in Chap. 7 we will show that every real x in [0, 1) can be written as

$$x = \sum_{i=1}^{\infty} \frac{d_i}{10^i} \quad \text{where the } d_i \text{ are in } \{0, 1, 2, \ldots, 9\}.$$

Hence, x is the limit (as n goes to infinity) of the sequence of rationals

$$\sum_{i=1}^{n} \frac{d_i}{10^i}.$$

We now turn to some of the properties of the reals. We start by the following definition.

Upper bound
A set A of numbers is said to be bounded above if there exists b such that

$$x \le b \quad \text{for all } x \in A.$$

The number b is then called an upper bound of A.

The next examples show that an upper bound of a set A may or may not be in A.

Example 1.6 Consider the interval $I = [0, 1)$. That is, I is the set of all reals larger than or equal to 0 and strictly less than 1. Hence, 1 is an upper bound of I. So are 2, 3, or any number larger than 1. Note that the upper bound 1 is not in I.

Example 1.7 Consider the interval $J = [0, 1]$. Again, 1 is an upper bound of J, but this time it is in J.

The following notion of *least upper bound* is crucial in analysis.

Least upper bound
A set A of numbers is said to have a least upper bound m if

(i) m is an upper bound of A;
(ii) if b is an upper bound of A, then $b \geq m$.

The number m is denoted by $\sup A$.

It will be shown in the exercises that a set of reals has at most one least upper bound.

Example 1.8 Consider the interval $I = [0, 1)$. We noted in Example 1.6 that 1 is an upper bound of I. We now show that is the *least* upper bound. Take any $0 \leq a < 1$ and let $b = \frac{a+1}{2}$. Then b is in I, and $b > a$. Hence, a cannot be an upper bound of I. Therefore, an upper bound of I must be larger than or equal to 1. Since 1 is an upper bound, it is the least upper bound.

The fundamental property of the real numbers is the following.

Fundamental property of the real numbers
There exists a set of numbers **R** called the set of *real* numbers and that contains the rational numbers. The set **R** has the following fundamental property. If A is a nonempty subset of **R** and is bounded above, then A has a least upper bound (which may or may not be in A).

A notion closely related, but not identical to, is the notion of maximum.

Maximum
A set A of numbers is said to have a maximum m if

(i) m belongs to A;
(ii) if a belongs to A, then $a \leq m$.

The number m is denoted by $\max A$.

We now state consequences of these definitions.

C1. If A has a maximum m, then m is also its least upper bound.

By the definition of m we have that $m \geq a$ for every a in A. So m is an upper bound of A. On the other hand, if b is an upper bound of A, then $b \geq m$ (since m is in A). Thus, m is the least upper bound of A.

C2. If A has a least upper bound, it is not necessarily a maximum.

In order to show C2, it is enough to give an example. Let $A = (0, 1)$, that is, A is the set of all real numbers strictly between 0 and 1. Note that 1 is an upper bound of A. If $m < 1$, then $(1 + m)/2 > m$, and $(1 + m)/2$ is in A (why?), so m is not an upper bound of A. Thus, the least upper bound of A is 1. But 1 is not a maximum of A (it does not belong to A). Moreover, if $m < 1$, it cannot be a maximum of A either since it is not an upper bound of A. So A has a least upper bound but no maximum.

We now turn to the equally important notion of lower bound.

Lower bound
A set A of numbers is said to be bounded below if there is m such that $m \leq a$ for every a in A. The number m is said to be a lower bound of A.
A set A is said to have a greatest lower bound m if

(i) m is a lower bound of A;
(ii) if b is a lower bound of A, then $b \leq m$.

The number m is denoted by $\inf A$.

The fundamental property of the reals may be stated in terms of the greatest lower bound instead of the least upper bound.

Greatest lower bound
If A is a nonempty subset of $\mathbf{R}$ and is bounded below, then A has a greatest lower bound.

In order to prove this, consider the set of the opposite numbers in A,

$$B = \{-a; a \in A\}.$$

Since A is nonempty, so is B. The set A has a lower bound m, so $a \geq m$ for every a in A. Therefore,

$$-a \leq -m$$

for every $-a$ in B. Thus, B is bounded above and is nonempty, so it has a least upper bound b. Since b is an upper bound for B, $-b$ is a lower bound for A (why?). Moreover, if m is a lower bound for A, $-m$ is an upper bound for B. Therefore, $-m \geq b$ (since b is the *least* upper bound of B), so $m \leq -b$. Thus, $-b$ is the greatest lower bound of A. We have proved that every nonempty set which is bounded below has a greatest lower bound.

The reader should note that things are a lot easier (and less interesting) when dealing with finite sets.

Finite sets
Let A be a finite nonempty set. That is, A can be written as

$$A = \{a_1, a_2, \dots, a_n\}$$

for some natural number n and some real numbers $a_1, a_2, \dots, a_n$. Then A has a minimum (which is also its greatest lower bound) and a maximum (which is also its least upper bound).

The properties above make sense intuitively: just order your finite set in increasing order. This can be proved by induction on the number of elements in the sets. Since this is tedious and not very instructive, we will omit this proof.

The following example is a typical application in analysis of least upper bounds.

Example 1.9 Assume that A has a least upper bound m. Show that there is an element a in A such that $a > m - 1/2$.

Observe that $m - 1/2$ cannot be an upper bound of A since $m - 1/2 < m$ and m is the *least* upper bound. Recall that, by definition, m is an upper bound of A if for every a in A, $a \le m$. Hence, $m - 1/2$ is not an upper bound of A means that there is at least one a in A such that $a > m - 1/2$. This proves our claim.

Example 1.10 The fundamental property does not hold for the rationals. Consider

$$A = \left\{r \in \mathbf{Q} : r > \sqrt{2}\right\}.$$

That is, A is the set of all rationals strictly larger than $\sqrt{2}$. The set A is not empty (2 is in A) and is bounded below by $\sqrt{2}$. Assume, by contradiction, that A has a greatest lower bound m *in* the rationals. If $m < \sqrt{2}$, then as we will see below (by the density property of the rationals), it is always possible to squeeze a rational between two real numbers. Hence, there is a rational q such that

$$\sqrt{2} > q > m.$$

Then q is a lower bound of A (why?) and is larger than the greatest lower bound m. We have a contradiction. Therefore, the rational m must be larger than or equal to $\sqrt{2}$. Since $\sqrt{2}$ is irrational, it must be strictly larger than $\sqrt{2}$. Again by the density of the rationals, we may find a rational s such that

$$\sqrt{2} < s < m.$$

This implies that s is in A and therefore contradicts the fact that m is a lower bound of A. We reach a contradiction again. Thus, there is no rational greatest lower bound of A. The fundamental property does not hold in the rationals. Observe, however, that by the fundamental property of the reals, A has a greatest lower bound in $\mathbf{R}$. It is $\sqrt{2}$, see the exercises.

We turn to another important property of the reals.

Archimedean property
For any real number a, there exists a natural number n such that $n > a$.

We do a proof by contradiction. Assume that the Archimedean property is not true. The negation of the Archimedean property is: there exists a real number a such that for all natural numbers n, we have $n \leq a$.

Thus, assuming that the Archimedean property fails, we see that $\mathbf{N}$ is bounded above by a real a. By the fundamental property of the reals, the nonempty set $\mathbf{N}$ has a least upper bound m. Hence, $m - 1$ is not an upper bound of $\mathbf{N}$. That is, there exists a natural n such that

$$n > m - 1,$$

and therefore,

$$n + 1 > m.$$

That is, $n + 1$ is a natural is larger than m, which is an upper bound of the naturals. We have our contradiction. The Archimedean property must hold.

Example 1.11 Consider the set

$$B = \left\{1 - \frac{1}{n} : n \in \mathbf{N}\right\}.$$

That is,

$$B = \{0, 1 - 1/2, 1 - 1/3, 1 - 1/4, \ldots\}.$$

Intuitively it is clear that 1 is the least upper bound of B. We will now prove this. First, observe that for every natural n, we have

$$1 - 1/n < 1.$$

Therefore, 1 is an upper bound of B. We now need to show that 1 is the least upper bound. To show this, we are going to pick any $m < 1$ and prove that m cannot be an upper bound of B. By the Archimedean property there is a natural n such that

$$n > \frac{1}{1 - m}.$$

Therefore, $1 - m > 1/n$ and $m < 1 - 1/n$. That is, we have found an element of B, $1 - 1/n$, which is larger than m. Hence, m is not an upper bound of B. We have a contradiction. Hence, the least upper bound of B is 1.

Density of the rational numbers in the reals
For any real numbers $a < b$, there exists a rational number between a and b.

In order to prove the density of the rationals in the reals, we start with the following:

Lemma *Let $x \geq 0$ and $\epsilon > 0$ be two reals. Then there exists an integer $n \geq 0$ such that*

$$n\epsilon \leq x < (n+1)\epsilon.$$

Consider the set

$$A = \{k \in \mathbf{N} : k\epsilon > x\}.$$

By the Archimedean property, there is a natural p such that

$$p > x/\epsilon.$$

This shows that A is a nonempty subset (p is in A) of the naturals. By the well-ordering principle, A has a minimum m. Therefore, $m-1$ is not in A. At this point there are two possibilities. Either $m = 1$ or $m \geq 2$. If $m = 1$, then $x < \epsilon$ (why?). Thus,

$$n\epsilon \leq x < (n+1)\epsilon$$

holds for $n = 0$.

On the other hand, if $m \geq 2$, then $m-1$ is a natural, and since it is not in A (why?), we have

$$(m-1)\epsilon \leq x.$$

Using that m is in A, we get

$$(m-1)\epsilon \leq x < m\epsilon.$$

That is,

$$n\epsilon \leq x < (n+1)\epsilon$$

for $n = m-1$, and the lemma is proved.

We now prove that the rationals are dense in the reals. Assume that $0 \leq a < b$. We need to find a rational between a and b. By the Archimedean property, there is a natural q such that

$$q > \frac{1}{b-a}.$$

Using the lemma with $\epsilon = \frac{1}{q}$, there exists a natural n such that

$$n\epsilon \leq a < (n+1)\epsilon.$$

That is,

$$\frac{n}{q} \leq a < \frac{n+1}{q}.$$

Let $r = \frac{n+1}{q}$; r is a rational, and $r > a$. From the inequality

$$q > \frac{1}{b-a}$$

we get

$$b > \frac{1}{q} + a \geq \frac{1}{q} + \frac{n}{q} = r.$$

Therefore, we have found a rational strictly between a and b. We have proved this in the particular case $0 \leq a < b$. The other cases (a and b negative, a negative and b positive) are easy consequences of this result and are left as exercises for the reader.

Density of the irrational numbers in the reals
For any real numbers $a < b$, there exists an irrational number between a and b.

This is not difficult using the density of the rationals in the reals and is proved in the exercises.

Our main motivation to introduce the real numbers was that $\sqrt{2}$ is not rational. The least we can do is show that $\sqrt{2}$ is a real number! We are actually going to show the following more general result.

Square roots
Let $a \geq 0$ be a real number. Then the equation $x^2 = a$ has exactly one positive solution in the reals. This positive solution is denoted by $\sqrt{a}$.

Note that $x^2 = 0$ has the unique solution $x = 0$. Hence, $\sqrt{0} = 0$. We now consider the case $a > 0$.

That the equation $x^2 = a$ has at most one solution is easy and is left as an exercise. The more difficult part is to show that there is at least one solution. Consider the set

$$A = \left\{x > 0 : x^2 < a\right\}.$$

Note that if $a < 1$, then $a^2 < a$, and a is in A. If $a > 1$, then $1^2 = 1 < a$, and 1 is in A. Hence, A is not empty and has at least one strictly positive number in it. Note that

$$(a+1)^2 = a^2 + 2a + 1 > a.$$

Hence, if x is in A, then $x^2 < a < (a+1)^2$. Thus, $x < a + 1$ (why?), and A is bounded above by $a + 1$ and is nonempty. According to the fundamental property of the reals, A has a least upper bound m. Since there are strictly positive numbers in A, we must have $m > 0$.

We now prove that $m^2 = a$, and thus m will provide the positive solution of the equation $x^2 = a$ we are looking for. In order to do so, we are going to show that we cannot have $m^2 < a$ or $m^2 > a$. We will first assume that $m^2 < a$ and show that this leads to a contradiction. For two numbers a and b, we use the notation $\min(a, b)$ to indicate the smallest of a and b. Let $\epsilon = \min(1, \frac{a-m^2}{2(2m+1)})$. Thus, ϵ is less than 1 but strictly larger than 0 since we are assuming that $m^2 < a$. We now compute

$$(m+\epsilon)^2 = m^2 + 2m\epsilon + \epsilon^2 \leq m^2 + 2m\epsilon + \epsilon = m^2 + (2m+1)\epsilon,$$

where the inequality $\epsilon^2 \leq \epsilon$ comes from the fact that $\epsilon \leq 1$. Now we use that

$$\epsilon \leq \frac{a-m^2}{2(2m+1)}$$

to get

$$(m+\epsilon)^2 \leq m^2 + (2m+1)\frac{a-m^2}{2(2m+1)} = m^2 + \frac{a-m^2}{2} = \frac{a+m^2}{2}.$$

Since we assume that $m^2 < a$, we have that $\frac{a+m^2}{2} < a$. Therefore, $m+\epsilon$ belongs to A, but that is impossible since it is larger than m, an upper bound of A. We have a contradiction. We cannot have $m^2 < a$.

Next, we show in a very similar way that the assumption $m^2 > a$ also leads to a contradiction. Set $\epsilon = \frac{m^2-a}{4m}$. We have

$$(m-\epsilon)^2 = m^2 - 2m\epsilon + \epsilon^2 > m^2 - 2m\epsilon = m^2 - 2m\frac{m^2-a}{4m} = \frac{m^2+a}{2} > a.$$

That is, by picking ϵ small enough we get that $(m-\epsilon)^2 > a$. Therefore, $m-\epsilon$ is an upper bound of A. But m is the least upper bound of A, so we have a contradiction. The only possibility left is $m^2 = a$. That is, we have found a solution of the equation $x^2 = a$.

Remark The fundamental property of the real numbers provides the *existence* of the number m, and then we prove that this number is actually a solution of $x^2 = a$. Many proofs in analysis are done using this pattern.

Exercises

1. Show that the equation $x^2 = 2$ has at most one positive solution.
2. Show that a set A has at most one least upper bound.
3. (a) Assume that the set A has a greatest lower bound m. Show that for every $\epsilon > 0$, there is an element a in A such that

$$m \leq a < m + \epsilon.$$

   (b) State and prove the analogous property for the least upper bound of a set A.
4. (a) Show that if a set has a minimum, it also has a greatest lower bound.
   (b) Show that the converse of (a) does not hold.
5. Give an example of a nonempty set A that has no greatest lower bound.
6. Let A be the set

$$\left\{\frac{1}{n} : n \in \mathbf{N}\right\}.$$

(a) Show that A has a greatest lower bound.
(b) Guess what the greatest lower bound is.
(c) Prove that your guess is correct.

7. Let $a > 0$ and b be real numbers. Prove that there is a natural n such that
$$na > b.$$
8. Suppose that $a < b$. Show that there is a natural n such that $b > a + 1/n$.
9. (a) Show that if a^2 is irrational, so is a.
(b) Is the converse of (a) true?
10. (a) Assume that $0 \le a < \epsilon$ for every $\epsilon > 0$. Show that $a = 0$.
(b) Let a and b such that for every $\epsilon > 0$, we have $a < b + \epsilon$. Show that $a \le b$.
11. Consider a set A and a real t such that if $b > t$, then b does not belong to A. Show that A has an upper bound.
12. We proved that there exists a rational between two real numbers $0 \le a < b$. Use this result to prove that for any $a < b$, there is at least a rational between a and b.
13. Show that there are infinitely many rational numbers between a and b.
14. In this exercise we continue Example 1.10. Let A be the following set of rationals:
$$A = \left\{ r \in \mathbf{Q} : r > \sqrt{2} \right\}.$$
(a) Show that A has a greatest lower bound m.
(b) Show that $m = \sqrt{2}$.
15. In this exercise we prove that for any $a < b$, there is an irrational number between a and b.
(a) Let q be a rational number strictly between a and b. Show that there exists a natural f such that $\sqrt{2} < f(b - q)$.
(b) Show that $q + \frac{\sqrt{2}}{f}$ is irrational. (Do a proof by contradiction.)
(c) Show that $q + \frac{\sqrt{2}}{f}$ is strictly between a and b.
16. We proved already that $\sqrt{2}$ is irrational. In this exercise we prove that for *all* naturals n, either $\sqrt{n}$ is a natural (when n is a perfect square), or it is an irrational. We do a proof by contradiction. Assume that $\sqrt{n}$ is a rational but not a natural, and we will find a contradiction.
(a) Use the lemma in this section to show that there is a natural k such that
$$k < \sqrt{n} < k + 1.$$
(b) Assuming that $\sqrt{n}$ is rational, show that there is a smallest natural ℓ such that $\ell\sqrt{n}$ is a natural.
(c) Show that
$$m = \ell\left(\sqrt{n} - k\right)$$
is a natural number strictly less than ℓ.
(d) Show that $m\sqrt{n}$ is a natural. Find the contradiction and conclude.

1.4 Power Functions

We start by stating easy but important inequalities.

Inequalities

I1. If $0 \le a < b$, then $a^n < b^n$ for any natural number n.
I2. If $0 < a < 1$, then $a^n < a$ for any natural number $n \ge 2$.
I3. If $a > 1$, then $a^n > a$ for any natural number $n \ge 2$.

We now prove these inequalities. We have

$$b^n - a^n = (b-a)\sum_{k=0}^{n-1} b^{n-1-k}a^k.$$

Note that all the terms in the sum are positive or 0. Hence, the sum is larger than its first term $b^{n-1} > 0$. That is,

$$\sum_{k=0}^{n-1} b^{n-1-k}a^k \ge b^{n-1}.$$

Thus,

$$b^n - a^n \ge (b-a)b^{n-1} > 0$$

for $b > a$. This proves I1.

To prove I2 and I3, note that

$$a^n - a = a\big(a^{n-1} - 1\big).$$

By I1, if $a < 1$, we have $a^{n-1} < 1^{n-1} = 1$, and so $a^n - a < 0$. This proves I2.

By I1, if $a > 1$, then $a^{n-1} > 1$, and so $a^n - a > 0$. This proves I3.

We define the nth root.

Roots

For any natural number n and any real number $a \ge 0$, there is a unique positive solution x of the equation $x^n = a$. We call the solution the nth root of a and denote it by $a^{1/n}$.

The strategy to prove[1] the existence and uniqueness of a positive solution of $x^n = a$ is the same as the one used to prove the result in the particular case $n = 2$ in the preceding section. We first use an algebraic identity to show the uniqueness, and then we prove the existence by using the fundamental property of the reals.

[1]This proof is a little long and may be omitted.

Lemma 1.1 *Assume that* $x \geq 0$, $y \geq 0$, *and* $x^n = y^n$ *for some natural number* n. *Then* $x = y$.

If $x^n = y^n = 0$, then $x = y = 0$ and we are done. If $x^n = y^n > 0$, since $x \geq 0$ and $y \geq 0$, we have $x > 0$ and $y > 0$ (why?). We have

$$x^n - y^n = (x - y)\sum_{k=0}^{n-1} x^{n-1-k}y^k = 0.$$

Since x and y are strictly positive, the sum in the right-hand side has only strictly positive terms. Thus, $x - y = 0$, and we have proved Lemma 1.1.

Lemma 1.1 implies that there is at most one solution to $x^n = a$. If $a = 0$, it is clear that there is only the solution $x = 0$. For the rest of the existence proof, we assume that $a > 0$. Define the set

$$A = \{y > 0 : y^n < a\}.$$

Note that since $a > 0$ we have $c = \frac{a}{a+1} < 1$ (why?) and $c > 0$. By I2 $c^n < c$. Note also that $c < a$ (why?), and so $c^n < c < a$. That is, c is in A, and A is nonempty.

Observe now that if $t > a + 1$, then by I1 $t^n > (a + 1)^n$. Moreover, by I3 $(a + 1)^n > a + 1 > a$, and so $t^n > a$. Thus, if $t > a + 1$, then t is not in A. Therefore, $a + 1$ is an upper bound of A. Since A is bounded above by $a + 1$ and nonempty, it has a least upper bound that we denote by m.

We want to show that $m^n = a$ and therefore that m is the unique solution of the equation $x^n = a$. We do a proof by contradiction. We will need the following:

Lemma 1.2 *Assume that* $0 < x < y$ *and that* $n \geq 2$ *is a natural. Then*

$$0 < y^n - x^n < (y - x)ny^{n-1}.$$

To prove Lemma 1.2, we start with

$$y^n - x^n = (y - x)\sum_{k=0}^{n-1} y^{n-1-k}x^k.$$

Since $0 < x < y$, if we replace x by y in each term of the sum, we get something bigger. For every integer k between 0 and $n - 1$, we have

$$x^k \leq y^k,$$

and the inequality is strict for $k \geq 1$. Hence,

$$y^{n-1-k}x^k \leq y^{n-1-k}y^k = y^{n-1}$$

and

$$y^n - x^n < (y - x)\sum_{k=0}^{n-1} y^{n-1} = (y - x)ny^{n-1},$$

where we use the fact that from $k = 0$ to $n - 1$ there are n identical terms in the sum above. This proves Lemma 1.2.

Assume now that $m^n < a$. Let $0 < \delta < 1$ be a real number. Using Lemma 1.2 with $y = m + \delta$ and $x = m$, we get

$$(m+\delta)^n - m^n < \delta n(m+\delta)^{n-1} < \delta n(m+1)^{n-1},$$

where the last inequality comes from $\delta < 1$ and I1. For two reals a and b, let $\min(a, b)$ denote the smallest of the two numbers. Set

$$\delta = \min\left(1, \frac{a-m^n}{2n(m+1)^{n-1}}\right),$$

which is positive since we are assuming that $a > m^n$. Thus,

$$\delta n(m+1)^{n-1} \leq \frac{a-m^n}{2n(m+1)^{n-1}} n(m+1)^{n-1} = \frac{a-m^n}{2}.$$

Hence,

$$(m+\delta)^n - m^n < \frac{a-m^n}{2},$$

that is,

$$(m+\delta)^n < \frac{a+m^n}{2} < a$$

since $a > m^n$. Thus, $(m+\delta)$ is in A but is larger than the upper bound m. This is a contradiction. We cannot have $a > m^n$.

We now prove that we cannot have $a < m^n$ either, and therefore we have $a = m^n$. Assume by contradiction that $a < m^n$. Let $0 < h < m$. Apply Lemma 1.2 to get

$$m^n - (m-h)^n < nhm^{n-1}.$$

Pick

$$h = \frac{m^n - a}{nm^{n-1}}.$$

Note that $h > 0$ since $m^n > a$. Note also that

$$h = \frac{m^n - a}{nm^{n-1}} < \frac{m^n}{nm^{n-1}} = \frac{m}{n} \leq m.$$

Hence, $h < m$. We have

$$nhm^{n-1} = nm^{n-1}\frac{m^n - a}{nm^{n-1}} = m^n - a.$$

Therefore,

$$m^n - (m-h)^n < m^n - a,$$

and so $(m-h)^n > a$. Observe that if $t > m - h$, then by I1 $t^n > (m-h)^n > a$, and t does not belong to A. No real above $m - h$ is in A. That is, $m - h$ is an upper bound of A. But m is supposed to be the *least* upper bound. We have a contradiction again. This concludes the proof that for any $a \geq 0$, the equation $x^n = a$ has a unique positive solution.

We now state some important rules for powers.

Rules for natural powers
For any real numbers $a > 0$, $b > 0$ and natural numbers s and t, we have:

P1. $a^{s+t} = a^s a^t$;
P2. $(a^s)^t = a^{st}$;
P3. $(ab)^s = a^s b^s$.

Recall our definition of a^n:

$$a^1 = a$$

and

$$a^{n+1} = a^n a \quad \text{for all } n \geq 1.$$

The rules P1, P2, and P3 can be proved by induction. For P1, we fix a natural s, and we do an induction on t. For $t = 1$, by definition we have

$$a^{s+1} = a^s a.$$

That is, the formula holds for $t = 1$. Assume that it holds for t. Then

$$a^{s+t+1} = a^{s+t} a = a^s a^t a = a^s a^{t+1},$$

where the second equality is the induction hypothesis. The other equalities come from our definition of power. Hence, P1 is proved by induction for a fixed s and any natural t. Since s can be any natural, this proves P1 for all naturals s and t.

For P2, we also fix s and do an induction on t. Note that

$$\left(a^s\right)^1 = a^s,$$

and the formula holds for $t = 1$. Assume that it holds for t. Let $c = a^s$. By definition,

$$\left(a^s\right)^{t+1} = c^{t+1} = c^t c = \left(a^s\right)^t a^s.$$

By the induction hypothesis,

$$\left(a^s\right)^t a^s = a^{st} a^s.$$

By P1 we have

$$a^{st} a^s = a^{st+s} = a^{s(t+1)},$$

and thus P2 holds for all s and t.

P3 may be proved by induction on s and is left as an exercise.

We now turn to rational powers. We have defined $3^{1/5}$ as being the unique positive solution of the equation $x^5 = 3$. But what does $3^{2/5}$ mean? We define

$$3^{2/5} = \left(3^2\right)^{1/5} = \left(3^{1/5}\right)^2.$$

We need to check that the second equality holds. We also want $3^{2/5}$ to be the same as $3^{4/10}$: a proper definition of 3^r for a rational r does not depend on the particular fraction representing r. All these questions are addressed below.

Rational powers
For any positive rational number $r = n/m$ (where n and m are natural integers) and any real number $a \geq 0$, we define

$$a^r = \left(a^n\right)^{1/m} = \left(a^{1/m}\right)^n.$$

If r is a negative rational number and $a > 0$, then a^r is defined as $1/a^{-r}$.

For the definition above to make sense, we need to check that

$$\left(a^n\right)^{1/m} = \left(a^{1/m}\right)^n$$

and that if $r = n/m = p/q$, then

$$\left(a^n\right)^{1/m} = \left(a^p\right)^{1/q}.$$

Let m and n be natural numbers, a a positive real number, and let $x = (a^{1/m})^n$ and $y = (a^n)^{1/m}$. Observe that $y^m = a^n$ (by definition of the mth root of a^n). On the other hand,

$$x^m = \left(\left(a^{1/m}\right)^n\right)^m = \left(\left(a^{1/m}\right)^m\right)^n = a^n,$$

where the second equality comes from P2: for positive integers n and m, $(b^n)^m = (b^m)^n = b^{nm}$. Thus, $x^m = y^m$, and by Lemma 1.1, $x = y$.

We now check that if $r = n/m = p/q$, then

$$\left(a^n\right)^{1/m} = \left(a^p\right)^{1/q}.$$

Let x be the right-hand side, and y the left-hand side. By the definition of the mth root of a^n we have

$$x^m = a^n$$

and by P2

$$\left(x^m\right)^q = x^{mq} = a^{nq}.$$

Similarly,

$$y^{mq} = a^{mp}.$$

Since n/m and p/q represent the same rational number, we have $nq = mp$, so $x^{mq} = y^{mq}$, and by Lemma 1.1, $x = y$.

Rules for rational powers
For any real number $a > 0$ and any rational numbers s and t, the rules P1, P2, and P3 hold. Moreover, we have

P4. $a^{s-t} = \frac{a^s}{a^t}$.

We will prove P1 for positive rational exponents. We will use this to prove P4 in the same case. This will allow us to prove P1 in all cases and then P4 in all cases. The proofs of P2 and P3 are easier and are left as exercises.

Let $x = a^{s+t}$ and $y = a^s a^t$. Assume that $s = m/n$ and $t = p/q$, where m, n, p, q are naturals. First note that

$$x = a^{\frac{mq+np}{nq}} = \left(a^{mq+np}\right)^{1/nq}$$

by the definition of a rational power. Thus, $x^{nq} = a^{mq+np}$. On the other hand,

$$y^{nq} = \left(a^s\right)^{nq} \left(a^t\right)^{nq}$$

using P3 for natural powers. Therefore,

$$y^{nq} = a^{mq} a^{np} = x^{nq},$$

where the last equality is P1 for natural powers. By Lemma 1.1, $x = y$, and P1 is proved for positive rational powers.

We now prove P4 for positive rationals s and t. Assume that $s > t$. Then by P1, for positive rationals, we have

$$a^{s-t} a^t = a^{s-t+t} = a^s.$$

Hence,

$$a^{s-t} = \frac{a^s}{a^t}.$$

We now turn to $s < t$. Then

$$a^{t-s} a^s = a^t$$

and

$$a^{t-s} = \frac{a^t}{a^s}.$$

Recall that by definition

$$a^{-r} = \frac{1}{a^r}$$

for all rationals r. Thus,

$$\frac{1}{a^{t-s}} = \frac{a^s}{a^t}$$

and

$$a^{s-t} = \frac{a^s}{a^t}.$$

This proves P4 for positive rationals s and t.

We now prove P1 for any rationals. Assume that $s > 0$ and $t > 0$. By P4, for positive rationals, we have

$$a^{s-t} = \frac{a^s}{a^t} = a^s a^{-t}.$$

This proves P1 when one rational exponent is positive and the other negative. When they are both negative, we use P1 for positive rational exponents to get

$$a^{-s-t} = \frac{1}{a^{s+t}} = \frac{1}{a^s a^t} = \frac{1}{a^s}\frac{1}{a^t} = a^{-s}a^{-t}.$$

This proves P1 when both exponents are negative, and P1 is proved for any two rational exponents. We use this to prove P4 in all cases. By P1, for any rationals s and t, we have

$$a^{s-t}a^t = a^{s-t+t} = a^s.$$

Hence,

$$a^{s-t} = \frac{a^s}{a^t}.$$

This proves P4 for any rational exponents.

We now turn to rational power functions. Recall that a function f from a set A to a set B is a relation between A and B such that for each element of A, f assigns exactly one element in B. The function is said to have an inverse f^{-1} if one can reverse the assignment. That is, f has an inverse if for every y in B, there is a unique solution x in A of the equation

$$f(x) = y.$$

If that is the case, we can define the inverse function f^{-1} by setting $f^{-1}(y) = x$ where x is the unique solution of $f(x) = y$. By the definition of the inverse function, we have, for every y in B,

$$f(x) = f\big(f^{-1}(y)\big) = y$$

and, for every x in A,

$$f^{-1}(y) = f^{-1}\big(f(x)\big) = x.$$

For any rational $r > 0$, the function $f(x) = x^r$ is defined on the positive reals. Let $r = p/q$ where p and q are positive integers. Then, for any $x \geq 0$, $x^r = (x^p)^{1/q}$ is the unique positive qth root of x^p. If $r < 0$, then $x^r = 1/x^{-r}$, which is also uniquely defined for any $x \geq 0$.

Let $r \neq 0$ be a rational, and let $y \geq 0$ be a real. Is there a unique solution to

$$x^r = y?$$

The answer, of course, is yes. Let $r = p/q$, where p and q are positive integers. Then, if $x = y^{1/r}$, we have

$$x = y^{q/p},$$

and so $x^p = y^q$ and $y = x^r$. Thus, we have found a solution. Moreover, this solution is unique (why?).

We now summarize our findings.

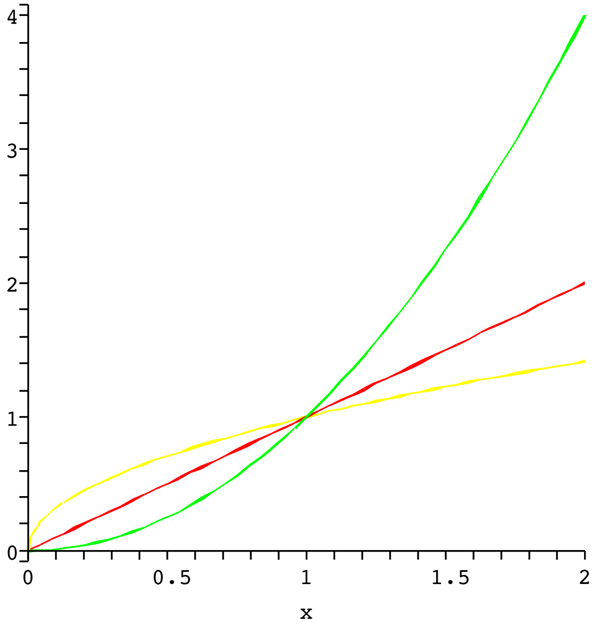

Fig. 1.1 These are the graphs of $y = \sqrt{x}$, $y = x$ and $y = x^2$

Power functions
Let $r \neq 0$ be a rational number. The function $x \to x^r$ is defined on the positive real numbers. It has an inverse function $x \to x^{1/r}$.

Figure 1.1 represents the graphs of $y = \sqrt{x}$, $y = x$, and $y = x^2$. A number of interesting facts are contained in this picture. In particular, if $r < 1$, then $x^r > x$ for $x < 1$ and $x^r < x$ for $x > 1$. If $r > 1$, then $x^r < x$ for $x < 1$ and $x^r > x$ for $x > 1$.

Increasing power functions
Let $r > 0$ be a rational number. The function $x \to x^r$ is strictly increasing on the positive real numbers. That is, for any real numbers $0 \leq a < b$,

$$a^r < b^r.$$

This is in fact an easy consequence of the same property for natural powers I1.

We first prove the property in the particular case $r = 1/n$. Assume that $0 < a < b$, and let n be a natural number. If $a^{1/n} \geq b^{1/n}$, then we would have, by I1,

$$\left(a^{1/n}\right)^n \geq \left(b^{1/n}\right)^n,$$

and so $a \geq b$. This is a contradiction. Therefore, $a^{1/n} < b^{1/n}$.

We are now ready for the general case. Let $r = m/n$ where m and n are natural numbers. Assume that $0 < a < b$. Then $a^{1/n} < b^{1/n}$ and $(a^{1/n})^m < (b^{1/n})^m$. This shows that $a^r < b^r$ and concludes the proof that the function $x \to x^r$ is increasing on the positive reals when $r > 0$.

We now give two more useful inequalities.

More Inequalities

I4 Let $0 < r < 1$ be a rational number, and $0 < a < 1$ be real number. Then $a^r > a$.

I5 Let $0 < r < 1$ be a rational number, and $a > 1$ be real number. Then $a^r < a$.

We start by proving I4. Let $r = p/q$ where $p < q$ with p and q natural numbers. Since $a < 1$, by I2,

$$a^{q-p} < a < 1.$$

Hence,

$$a^{q-p}a^p < a^p,$$

and so $a^q < a^p$. Using that $x \to x^{1/q}$ is strictly increasing, we get

$$\left(a^q\right)^{1/q} < \left(a^p\right)^{1/q}.$$

That is,

$$a < a^r.$$

This proves I4. The proof of I5 is similar and is left as an exercise.

We now have a mathematical definition for rational powers. But what does $3^{\sqrt{2}}$ mean? When we will introduce the exponential and logarithmic function in Chap. 3, we will define, for $a > 0$ and any real x,

$$a^x = \exp(x \ln a).$$

In particular,

$$3^{\sqrt{2}} = \exp\left(\sqrt{2} \ln 3\right).$$

Exercises

1. Solve the inequality $x^2 < x$.
2. Solve the inequality $x^3 > x^2$.
3. Show that for any real a, $\sqrt{a^2} = |a|$. (Use that $\sqrt{b}$ is the unique positive solution of the equation $x^2 = b$.)
4. Show that if x and y are positive reals and $x^r = y^r$ for some rational $r \neq 0$, then $x = y$.

5. (a) Prove P3 for natural powers.
   (b) Prove P2 for rational numbers.
   (c) Prove P3 for rational numbers.
6. Assume that $a > 1$. Show that for natural numbers n and m, $a^n > a^m$ if and only if $n > m$.
7. Assume that $a > 1$ is real and that the rationals s and t are such that $s < t$. Prove that $a^s < a^t$. (Raise both quantities to the appropriate power and use Exercise 6.)
8. Show that if the rational number r is negative, then the function $x \to x^r$ is decreasing on the positive reals. That is, show that if $0 \leq a < b$, then $a^r > b^r$.
9. Prove I5.
10. Assume that the rational number $r > 1$.
    (a) Show that if $a > 1$, then $a^r > a$.
    (b) Show that if $a < 1$, then $a^r < a$.
11. (a) Prove that $1 < \sqrt{2} < 2$.
    (b) Find two rationals r_1 and r_2 such that $r_1 < \sqrt{2} < r_2$ and $r_2 - r_1 < 1/100$.
12. Show that $x \to \frac{1}{\sqrt{1+x^2}}$ is a decreasing function on the positive reals.

Chapter 2
Sequences and Series

2.1 Sequences

A sequence is a function from the positive integers (possibly including 0) to the reals. A typical example is $a_n = 1/n$ defined for all integers $n \geq 1$. The notation a_n is different from the standard notation for functions $a(n)$ but means the same thing. We now define convergence for sequences.

Convergent sequence
The sequence a_n is said to converge to a real number a if for every $\epsilon > 0$, there is a natural number N (that usually depends on ϵ) such that if $n \geq N$, then

$$|a_n - a| < \epsilon.$$

a_n converges to a is denoted by $\lim_{n\to\infty} a_n = a$. The number a is called the limit of the sequence a_n.

Note that for sequences, one looks only at the limit as n goes to infinity. A sequence that does not converge is said to diverge. We will now study several examples.

Example 2.1 Consider a constant sequence $a_n = c$ for all $n \geq 1$, where c is a constant. Then, of course, a_n converges to c: take any $\epsilon > 0$, take $N = 1$; then for every $n \geq N$, we have $|a_n - c| = 0 < \epsilon$. Therefore, by definition, a_n converges to c.

Example 2.2 Let $a_n = 1/n$. Intuitively, it is clear that a_n converges to 0. We now prove this. Let $\epsilon > 0$. By the Archimedean property there is a natural $N > 1/\epsilon$. For $n \geq N$, we have $1/n \leq 1/N < \epsilon$ (why?). Thus,

$$1/n = |a_n - 0| < \epsilon \quad \text{for all } n \geq N,$$

and we have proved that a_n converges to 0.

R.B. Schinazi, *From Calculus to Analysis*,
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Example 2.3 Assume that the sequence a_n converges to ℓ. Let $b_n = a_{n+1}$. Let us show that b_n converges to the same ℓ.

Since a_n converges to ℓ, for any $\epsilon > 0$, there is a natural N such that if $n \geq N$, then $|a_n - \ell| < \epsilon$. Note that if $n \geq N$, then $n + 1 \geq N$ and $|a_{n+1} - \ell| < \epsilon$ as well. That is, if $n \geq N$, we have $|b_n - \ell| < \epsilon$. This proves that b_n converges to ℓ as well.

In order to prove that a sequence converges to 0, the following is sometimes useful.

Convergence to 0
A sequence a_n converges to 0 if and only if the sequence $|a_n|$ converges to 0.

Assume first that a_n converges to 0. For any $\epsilon > 0$, there is N such that if $n \geq N$, then

$$|a_n - 0| < \epsilon.$$

We have that

$$|a_n - 0| = |a_n| = \big||a_n| - 0\big|.$$

Hence,

$$\big||a_n| - 0\big| < \epsilon$$

for all $n \geq N$. This proves that $|a_n|$ converges to 0.

Assume now that $|a_n|$ converges to 0. For any $\epsilon > 0$, there is N such that if $n \geq N$, then

$$\big||a_n| - 0\big| < \epsilon.$$

But $||a_n| - 0| = |a_n - 0|$. Thus,

$$|a_n - 0| < \epsilon$$

for all $n \geq N$. This proves that a_n converges to 0 and completes the proof.

Example 2.4 Assume that a_n converges to ℓ, and let c be a constant. Then ca_n converges to $c\ell$.

If $c = 0$, then ca_n is the constant sequence 0, and it converges to $0 = 0 \cdot \ell$. So the property holds when $c = 0$. Assume now that $c \neq 0$. Let $\epsilon > 0$; since a_n converges to ℓ, there is a natural N such that if $n \geq N$, we have

$$|a_n - \ell| < \frac{\epsilon}{|c|}.$$

Multiplying across this inequality by $|c|$, we get

$$|ca_n - c\ell| < \epsilon.$$

This proves that ca_n converges to $c\ell$, and we are done.

The next example shows that not all oscillating sequences diverge.

Example 2.5 Let $a_n = (-1)^n/n$. Show that a_n converges to 0.

Observe that $|a_n| = 1/n$ converges to 0. Thus, a_n converges to 0.

We now turn to the notion of bounded sequence

Bounded sequences
A sequence a_n is said to be bounded if there exists a real number K such that

$$|a_n| < K \quad \text{for all } n.$$

There is an important relation between bounded sequences and convergent sequences.

A convergent sequence is bounded
If a sequence converges, then it must be bounded.

We now prove that a convergent sequence is bounded. Assume that the sequence a_n converges to some limit a. Take $\epsilon = 1$; since a_n converges to a, there is a natural N such that if $n \geq N$, then $|a_n - a| < \epsilon = 1$. Thus, by the triangle inequality we have

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a| \quad \text{for all } n \geq N.$$

The inequalities above show that the sequence is bounded for $n \geq N$. We now take care of $n < N$. We know that a finite set of real numbers always has a maximum and a minimum. Thus, let

$$M = \max\left\{|a_1|, |a_2|, \ldots, |a_{N-1}|\right\},$$

and let $K = \max(M, 1 + |a|)$. Then, we claim that

$$|a_n| < K \quad \text{for all } n \geq 1.$$

For if $n < N$, then $|a_n| < M \leq K$, while if $n \geq N$, then $|a_n| < |a| + 1 \leq K$. This completes the proof that a convergent sequence is bounded.

Example 2.6 Let c such that $|c| > 1$. Consider the sequence $b_n = c^n$ for $n \geq 0$. Let us show that b_n does not converge.

Let $|c| = 1 + a$ where $a = |c| - 1 > 0$. By Bernoulli's inequality we have

$$|b_n| = |c|^n = (1 + a)^n \geq 1 + na.$$

Note that the sequence $1 + na$ is not bounded (why?), and therefore b_n is not bounded either. Therefore, it cannot converge.

A bounded sequence does not necessarily converge. In order to show that a sequence does not converge, the following notion of subsequence is quite useful.

Subsequences
Let $1 \leq j_1 < j_2 < \cdots < j_n < \cdots$ be a strictly increasing sequence of natural numbers. Let a_n be a sequence of real numbers. Then a_{j_n} defines a new sequence of real numbers (it is in fact the composition of sequences a_n and j_n) and is called a subsequence of a_n. The name comes from the fact that all terms a_{j_n} are in the original sequence a_n.

Useful examples of strictly increasing sequences of natural numbers are $j_n = n$, $j_n = 2n$, $j_n = 2n + 1$, and $j_n = 2^n$.

Subsequences and convergence
A sequence a_n converges to a if and only if all the subsequences of a_n converge to a.

The proof is easy. First, the direct implication. Assume that a_n converges to a, and let a_{j_n} be a subsequence of a_n. For any $\epsilon > 0$, there is N such that if $n \geq N$, then $|a_n - a| < \epsilon$. It is easy to show (see the exercise) that $j_n \geq n$ for all $n \geq 1$. Thus, if $n \geq N$, then $j_n \geq n \geq N$ and $|a_{j_n} - a| < \epsilon$. Therefore, a_{j_n} converges to a. This proves that any subsequence of a_n converges to a.

We now prove the converse. We assume that all subsequences of a_n converge to a. But a_n is a subsequence of itself (take $j_n = n$), and thus a_n converges to a and we are done.

We use the preceding result in the next example.

Example 2.7 Let $a_n = (-1)^n$. Intuitively, it is clear that a_n does not converge (it oscillates between -1 and 1). We prove this. Note that the subsequence a_{2n} is the constant sequence 1 and so converges to 1. On the other hand, a_{2n+1} is the constant sequence -1 and so converges to -1. That is, we have found two subsequences that converge to two distinct limits. This proves that $(-1)^n$ does not converge.

The following principle is sometimes quite useful.

Squeezing principle
Assume that the three sequences a_n, b_n, and c_n are such that

$$a_n \leq b_n \leq c_n$$

and that the sequences a_n and c_n converge to the *same* limit ℓ. Then b_n converges to ℓ as well.

The result above not only proves the convergence of b_n, but it also gives the limit. We now prove it. Recall that

$$|x| < a \iff -a < x < a.$$

We start by writing that a_n and c_n converge to ℓ. For any $\epsilon > 0$, there are naturals N_1 and N_2 such that if $n \geq N_1$, then

$$|a_n - \ell| < \epsilon \iff \ell - \epsilon < a_n < \ell + \epsilon,$$

and if $n \geq N_2$, then

$$|c_n - \ell| < \epsilon \iff \ell - \epsilon < c_n < \ell + \epsilon.$$

Define $N = \max(N_1, N_2)$, so that the two double inequalities above hold for $n \geq N$. Thus, for $n \geq N$, using that b_n is squeezed between a_n and c_n, we have

$$\ell - \epsilon < a_n \leq b_n \leq c_n < \ell + \epsilon.$$

But this implies that

$$\ell - \epsilon < b_n < \ell + \epsilon \iff |b_n - \ell| < \epsilon \quad \text{for all } n \geq N.$$

That is, b_n converges to ℓ, and the squeezing principle is proved.

Next we give an application of the squeezing principle.

Example 2.8 Assume that a_n converges to 0 and that b_n is bounded. Then $a_n b_n$ converges to 0.

Since b_n is bounded, there is B such that $|b_n| < B$ for all n. Then,

$$0 \leq |a_n b_n| \leq B|a_n|.$$

That is, the sequence $|a_n b_n|$ is squeezed between the constant sequence 0 and the sequence $B|a_n|$. But they both converge to 0 (why?). By the squeezing principle, $|a_n b_n|$ converges to 0, and so does $a_n b_n$.

Example 2.9 Consider the sequence $a_n = n^r$ where r is a rational number. Discuss the convergence of a_n in function of r.

If $r = 0$, then a_n is the constant sequence 1, and it converges to 1.

Assume that $r > 0$. By contradiction, assume that the sequence $a_n = n^r$ is bounded. Then there is $B > 0$ such that for all n, $n^r < B$. Since $1/r > 0$, the function $x \to x^{1/r}$ is increasing, and we have $n < B^{1/r}$ for all naturals n. This contradicts the Archimedean property and shows that a_n cannot be bounded. Hence, if $r > 0$, a_n does not converge.

Let $r > 0$. We now show that n^{-r} converges to 0. Take any $\epsilon > 0$ and let $N > 1/\epsilon^{1/r}$; then for $n \geq N$, we have

$$\left|n^{-r} - 0\right| = 1/n^r \leq 1/N^r < \epsilon.$$

This proves that n^{-r} converges to 0.

The following is a useful property of least upper bounds.

Least upper bound and sequences
Let A be a subset in the reals with a least upper bound m. There exists a sequence a_n in A that converges to m.

The important part of this property is that the sequence is *in A*. The same property holds for the greatest lower bound, and the proof is left to the reader.

We now prove the property. Since m is the least upper bound of A, $m - 1$ is not an upper bound of A. Thus, there is at least one a in A such that $a > m - 1$. We pick one such a, and we denote it by a_1. Similarly, $m - 1/2$ cannot be an upper bound of A, so there is at least one a in A such that

$$a > m - 1/2.$$

We pick one such a, and we call it a_2. More generally, for every natural n, $m - 1/n$ is not an upper bound of A, and we may pick a_n in A such that

$$a_n > m - 1/n.$$

Hence, there exists a sequence a_n in A such that $a_n > m - 1/n$ for all $n \geq 1$. On the other hand, since the sequence a_n is in A, we must have $a_n \leq m$ for all $n \geq 1$. Thus,

$$m - 1/n < a_n \leq m.$$

It is easy to prove that $m - 1/n$ converges to m (it is also a consequence of the operations on limits to be proved in the next section). Hence, by the squeezing principle, the sequence a_n converges to m.

Exercises

1. (a) Let a_n be a sequence of reals converging to ℓ. Let $b_n = a_{n-1}$. Show that b_n converges to ℓ as well.
   (b) State a generalization of the result in (a) and prove your claim.
2. Let $1 \leq j_1 < j_2 < \cdots < j_n < \cdots$ be a sequence of natural numbers. Prove (by induction) that $j_n \geq n$ for all $n \geq 1$.
3. (a) Show that if a_n converges to a, then $|a_n|$ converges to $|a|$ (use that $||x| - |y|| \leq |x - y|$).
   (b) Is it true that if $|a_n|$ converges, then a_n converges? Prove it or give a counterexample.
4. (a) Assume that the real a is such that $|a| < \epsilon$ for any $\epsilon > 0$. Prove that $a = 0$.
   (b) Prove that a limit is unique (assume that a sequence a_n has two limits a and b, and show that $|a - b| < \epsilon$ for any $\epsilon > 0$.)
5. Assume that a_n converges to a. Prove that for any $\epsilon > 0$, a_n is in $(a - \epsilon, a + \epsilon)$ for all n except possibly finitely many.
6. Assume that a_n converges to 1.
   (a) Show that there is N such that if $n \geq N$, then $a_n < 2$.

(b) Show that there is N_1 such that if $n \geq N_1$, then $a_n > 1/2$.

7. Assume that the three sequences a_n, b_n, and c_n are such that
$$a_n \leq b_n \leq c_n \quad \text{for } n \geq 1000$$
and that the sequences a_n and c_n converge to the *same* limit ℓ. Explain how to modify the proof of the squeezing principle so that the conclusion still holds.

8. Let A be a subset in the reals with a greatest lower bound k. Show that there exists a sequence a_n in A that converges to k.

9. Assume that a_n is positive and converges to a limit ℓ.
   (a) Assume that $\ell = 0$. Show that $\sqrt{a_n}$ converges to 0.
   (b) Assume that $\ell > 0$. Show that $\sqrt{a_n}$ converges to $\sqrt{\ell}$.
$$\left(\text{Write that } \left|\sqrt{a_n} - \sqrt{\ell}\right| = \frac{|a_n - \ell|}{\sqrt{a_n} + \sqrt{\ell}}.\right)$$

10. Let a_n be a sequence of reals, and ℓ be a real. Let $b_n = |a_n - \ell|$. Show that a_n converges to ℓ if and only if b_n converges to 0.

11. Assume that a_n takes values in the set $\{0, 1, 2\}$ for every $n \geq 1$. That is, a_n can take only the values 0, 1, or 2. Suppose, moreover, that a_n converges to 1. Show that there is a natural N such that if $n \geq N$, then $a_n = 1$.

12. Consider a sequence a_n that takes values in the naturals.
   (a) Give an example of such a sequence.
   (b) Show that a_n converges if and only if it is stationary, that is, if and only if there is a natural N such that if $n \geq N$, then $a_n = a_N$.

13. Consider the sequence $a_n = n^{1/n}$. Let $b_n = a_n - 1$.
   (a) Show that $b_n > 0$ for $n \geq 2$.
   (b) Show that
$$n = (b_n + 1)^n \geq \frac{n(n-1)}{2} b_n^2.$$
   (Use the binomial expansion
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$
   where all $\binom{n}{k}$ are natural numbers and $\binom{n}{2} = n(n-1)/2$.)
   (c) Use (a) and (b) to get
$$0 < b_n \leq \sqrt{\frac{2}{n-1}}.$$
   (d) Show that $n^{1/n}$ converges to 1.

14. (a) Assume that a_n does not converge to a. Show that there exists an $\epsilon > 0$ and a subsequence a_{j_n} such that
$$|a_{j_n} - a| > \epsilon \quad \text{for all } n.$$
   (b) Let b_n be a sequence of reals. Assume that there exists a b such that any subsequence of b_n has a further subsequence that converges to b. Show that b_n converges to b. (Do a proof by contradiction and use (a).)

2.2 Monotone Sequences, Bolzano–Weierstrass Theorem, and Operations on Limits

We start with the notion of monotone sequence.

Monotone sequences
A sequence a_n is said to be increasing if for every natural n, we have $a_{n+1} \geq a_n$. A sequence a_n is said to be decreasing if for every natural n, we have $a_{n+1} \leq a_n$. A sequence which is increasing or decreasing is said to be monotone.

The sequence $a_n = 1/n$ is decreasing. The sequence $b_n = n^2$ is increasing, and the sequence $c_n = (-1)^n$ is not monotone. The following notions are crucial for monotone sequences.

Bounded below and above
A sequence a_n is said to be bounded below if there is a real number m such that $a_n > m$ for all n. A sequence a_n is said to be bounded above if there is a real number M such that $a_n < M$ for all n.

We are now ready to state an important convergence criterion for monotone sequences.

Convergent monotone sequences
An increasing sequence converges if and only if it is bounded above. A decreasing sequence is convergent if and only if it is bounded below.

We prove the statement about increasing sequences, the other one is similar and is left as an exercise. Assume that the sequence a_n converges. Then we know that it must be bounded and therefore bounded above. This proves one implication.

For the other implication, assume that the increasing sequence a_n is bounded above by a number M. Let A be the set

$$A = \{a_n, n \in \mathbf{N}\}.$$

The set A has an upper bound M and is not empty. The fundamental property of the reals applies: there is a least upper bound ℓ for A. We are now going to show that the sequence a_n converges to ℓ. Take any $\epsilon > 0$; since ℓ is the *least* upper bound, $\ell - \epsilon$ cannot be an upper bound. That is, there is an element in A strictly larger than ℓ. Therefore, there is N such that $a_N > \ell - \epsilon$. Note that up to this point we have not used that a_n is an increasing sequence. We now do. Take $n \geq N$. Then

$$a_n \geq a_N > \ell - \epsilon.$$

Hence,

$$\ell - a_n < \epsilon.$$

Since ℓ is larger than all a_n, we have

$$|a_n - \ell| = \ell - a_n < \epsilon \quad \text{for all } n \geq N.$$

That is, a_n converges to ℓ.

Example 2.10 Let $a > 1$. We show that $a_n = a^{1/n}$ converges and find its limit.

Since $\frac{1}{n+1} < \frac{1}{n}$ and $a > 1$, we have that $a_{n+1} < a_n$. That is, a_n is decreasing. Moreover, we have $a^{1/n} > 1^{1/n} = 1$ (the function $x \to x^{1/n}$ is increasing). Therefore, a_n is decreasing and bounded below by 1, and thus it converges.

We are now going to find the limit ℓ of a_n. Consider the subsequence a_{2n}. It must converge to ℓ as well. On the other hand $a_{2n} = a^{\frac{1}{2n}} = (a_n)^{1/2}$ converges to $\ell^{1/2}$ (see Exercise 9 in Sect. 2.1). Therefore, $\ell = \ell^{1/2}$. Either $\ell = 0$ or $\ell = 1$, but ℓ cannot be 0 (why not?), therefore it is 1.

Example 2.11 Let c be in (0, 1) and define $a_n = c^n$. We show that c_n converges to 0.

Since $c < 1$, we have $c^n c < c^n$. That is, $a_{n+1} < a_n$. The sequence a_n is strictly decreasing. It is also a positive sequence, and therefore it is bounded below by 0. Hence, the sequence a_n converges to some ℓ. We know that a_{n+1} converges to the same limit ℓ. However, $a_{n+1} = ca_n$, and since a_n converges to ℓ, we know that ca_n converges to $c\ell$. Hence, a_{n+1} converges to ℓ and to $c\ell$. We need to have $\ell = c\ell$. Therefore, either $c = 1$ (but we know that $c < 1$) or $\ell = 0$. Hence, $\ell = 0$.

As we have seen, not all bounded sequences converge. However, the following weaker statement holds and is very important.

Bolzano–Weierstrass Theorem
A bounded sequence has always a convergent subsequence.

Bolzano–Weierstrass is one of the fundamental theorems in analysis. We will apply it in Chap. 5, for instance, to prove the extreme value theorem.

In order to prove this theorem, we first show that every sequence (bounded or not) has a monotone subsequence.

Lemma *Every sequence has a monotone subsequence.*

We prove this lemma. Consider a sequence a_n. Let

$$A = \{m \in \mathbf{N} : \text{ for all } n > m, \ a_n \leq a_m\}.$$

There are three possibilities: A may be empty, finite, or infinite. Assume first that A is finite. It has a maximum N (this is true for any finite set). Set $n_1 = N + 1$. Since

n_1 is not in A, there exists at least one $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$. Assume that we have found $n_1 < n_2 < \cdots < n_k$ such that

$$a_{n_1} < a_{n_2} < \cdots < a_{n_k}.$$

Using that n_k is not in A, there is $n_{k+1} > n_k$ such that

$$a_{n_{k+1}} > a_{n_k}.$$

That is, we may construct inductively a strictly increasing sequence when A is finite. Note that in case A is empty, the same construction works for $N = 1$.

Assume now that A is infinite. Since A is a nonempty set of naturals, it has a minimum n_1. Let $n_2 > n_1$ be another element in A. Since n_1 is in A, we have

$$a_{n_2} \leq a_{n_1}.$$

Assume that we have found $n_1 < n_2 < \cdots < n_k$ in A such that

$$a_{n_1} \geq a_{n_2} \geq \cdots \geq a_{n_k}.$$

Since A is infinite, there is n_{k+1} in A such that $n_{k+1} > n_k$. Using that n_k is in A, we have

$$a_{n_{k+1}} \leq a_{n_k}.$$

That is, we may construct inductively a decreasing sequence when A is infinite. This completes the proof of the lemma.

It is now very easy to prove the Bolzano–Weierstrass theorem. Let a_n be a bounded sequence. According to the lemma, a_n has a monotone subsequence a_{n_k}. Since a_n is bounded, so is a_{n_k}. Hence, a_{n_k} is a monotone and bounded sequence. Therefore, it converges. This completes the proof of the theorem.

Example 2.12 Show that if a_n is a sequence in [0, 1], then it has a subsequence that converges.

Since for all $n \geq 1$, a_n is in [0, 1], we have $|a_n| \leq 1$. Hence a_n is bounded, and by Bolzano–Weierstrass, it has a convergent subsequence.

We now turn to the operations on limits.

Operations on limits

Assume that the sequences a_n and b_n converge, respectively, to a and b. Then

(i) $\lim_{n\to\infty}(a_n + b_n) = a + b$.
(ii) $\lim_{n\to\infty} a_n b_n = ab$.
(iii) Assume that b_n is never 0 and that b is not 0. Then

$$\lim_{n\to\infty} a_n/b_n = a/b.$$

(iv) Assume that for all n, $a_n \leq b_n$. Then $a \leq b$.

We prove the statements above. Since a_n and b_n converge, for any $\epsilon > 0$, there are natural numbers N_1 and N_2 such that if $n \geq N_1$, we have

$$|a_n - a| < \epsilon/2,$$

and if $n \geq N_2$, we have

$$|b_n - b| < \epsilon/2.$$

Take $N = \max(N_1, N_2)$ so that both statements above hold for $n \geq N$ and use the triangle inequality to get

$$\big|(a_n + b_n) - (a + b)\big| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves (i).

We now deal with (ii). The case $b = 0$ can be dealt with by using the result in Example 2.8, Sect. 2.1. Since a_n converges, it is bounded. We proved already that the product of a bounded sequence and a sequence converging to 0 converges to 0.

We now assume that $b \neq 0$. We have

$$|a_n b_n - ab| = \big|a_n(b_n - b) + a_n b - ab\big| \leq |a_n||b_n - b| + |b||a_n - a|. \tag{2.1}$$

Since a_n converges, it is bounded, so there is $A > 0$ such that $|a_n| < A$ for every n. Take $\epsilon > 0$; there is N_1 such that if $n \geq N_1$, then

$$|a_n - a| < \frac{\epsilon}{2b}.$$

There is also N_2 such that if $n \geq N_2$, then

$$|b_n - b| < \frac{\epsilon}{2A}.$$

We use the two preceding inequalities in (2.1) to get that if $n \geq \max(N_1, N_2)$,

$$|a_n b_n - ab| \leq |a_n||b_n - b| + |b||a_n - a| \leq A\frac{\epsilon}{2A} + |b|\frac{\epsilon}{2b} = \epsilon.$$

This completes the proof of (ii).

To prove (iii), it is enough to prove that $1/b_n$ converges to $1/b$. We can then use (ii) to show that $a_n/b_n = a_n(1/b_n)$ converges to $a(1/b) = a/b$. We start with

$$|1/b_n - 1/b| = \left|\frac{b_n - b}{b_n b}\right|.$$

The only difficulty is to show that b_n is bounded away from 0. Since b_n converges to b, there is a natural number N_1 such that if $n \geq N_1$, then

$$|b_n - b| \leq |b|/2,$$

where we are using the definition of convergence with $\epsilon = |b|/2 > 0$ (since $b \neq 0$). By the triangle inequality we get, for $n \geq N_1$,

$$|b| = |b - b_n + b_n| \leq |b - b_n| + |b_n| < |b|/2 + |b_n|.$$

That is,

$$|b_n| > |b|/2 \quad \text{for all } n \geq N_1.$$

In particular, we have

$$\frac{1}{|b_n b|} < \frac{2}{b^2} \quad \text{for all } n \geq N_1.$$

On the other hand, since b_n converges to b, for any $\epsilon > 0$, there is N_2 such that if $n \geq N_2$, then

$$|b_n - b| < \frac{\epsilon b^2}{2}.$$

Therefore, for $n \geq \max(N_1, N_2)$, we have

$$|1/b_n - 1/b| = \left|\frac{b_n - b}{b_n b}\right| = \frac{|b_n - b|}{|b_n b|}| < \frac{\epsilon b^2}{2}\frac{2}{b^2} = \epsilon.$$

That is, $1/b_n$ converges to $1/b$, and the proof of (iii) is complete.

For (iv), we do a proof by contradiction. Assume that $a > b$ and let $\epsilon = \frac{a-b}{2} > 0$. Since a_n converges to a, there exists N_1 such that if $n \geq N_1$, then

$$|a_n - a| < \epsilon.$$

In particular, $a_n > a - \epsilon = \frac{a+b}{2}$ if $n \geq N_1$. On the other hand, since b_n converges to b, there exists N_2 such that if $n \geq N_2$, then

$$|b_n - b| < \epsilon.$$

In particular, $b_n < b + \epsilon = \frac{a+b}{2}$ if $n \geq N_2$. Thus, if $n > \max(N_1, N_2)$, then

$$b_n < \frac{a+b}{2} < a_n,$$

contradicting the assumption $a_n \leq b_n$ for all n. This completes the proof of (iv).

Example 2.13 Assume that a_n converges to a and that $a_n + b_n$ converges to c. We show that b_n converges.

We start by writing

$$b_n = (b_n + a_n) + (-a_n).$$

We have assumed that $b_n + a_n$ converges to c. By (ii), $(-a_n)$ converges to $-a$ (why?). Hence, by (i), $b_n = (b_n + a_n) + (-a_n)$ converges to $c - a$. This completes the proof.

Exercises

1. Show that if a_n and b_n converge to a and b, respectively, then $ca_n + db_n$ converges for any constants c and d.
2. Assume that a_n is increasing. Show that if $n > m$, then $a_n \geq a_m$.
3. Prove that a decreasing sequence converges if and only if it is bounded below.
4. Assume that a_n is an increasing sequence, b_n is a decreasing sequence, and $a_n \leq b_n$ for all $n \geq 1$.
   (a) Show that a_n and b_n converge.

(b) If in addition to the previous hypotheses, we have that $b_n - a_n$ converges to 0, prove that a_n and b_n converge to the same limit.

5. Assume that $a_n < b_n$ for all naturals n. Assume that a_n and b_n converge to a and b, respectively. Is it true that $a < b$? Prove it or give a counterexample.
6. Assume that a_n converges to ℓ and is bounded below by m. Show that $\ell \geq m$.
7. Show that for any $a > 0$, $a^{1/n}$ converges to 1 (you may use the fact that the result has been proved for $a > 1$ in Example 2.10).
8. Suppose that a_n is increasing. Show that a_n converges if and only if it has a subsequence which converges.
9. Assume that a_n converges to $a \in (0, 1)$.
   (a) Show that there is N such for $n \geq N$, we have a_n in $(0, 1)$.
   (b) Is the statement in (a) true if we replace $(0, 1)$ by $[0, 1)$?
10. We define a sequence a_n recursively by setting $a_1 = 2$ and

$$a_{n+1} = \frac{2a_n - 1}{a_n}.$$

   (a) Compute the first five terms of this sequence.
   (b) Show that for all $n \geq 1$, if $a_n > 1$, then $a_{n+1} > 1$.
   (c) Use (b) to show that the sequence a_n is well defined and that a_n is bounded below by 1.
   (d) Show that a_n is decreasing.
   (e) Show that a_n converges to some limit $\ell \geq 1$.
   (f) Show that

$$\frac{2a_n - 1}{a_n} \quad \text{converges to } \frac{2\ell - 1}{\ell}.$$

   (g) Find ℓ.
11. Assume that $a_n - a$ converges to 0. Show that a_n converges to a.
12. Assume that $|a_n - a| \leq 1/n$ for all $n \geq 1$. Show that a_n converges to a.
13. Assume that $a_n - b_n$ converges and that b_n converges. Show that a_n converges.
14. Consider a sequence a_n that converges to $a > 0$. Show that there is a natural N such that if $n \geq N$, then $a_n > 0$.
15. Assume that $|c| < 1$. Show that the sequence c^n converges to 0.
16. A set C in $\mathbf{R}$ is said to be closed if for every convergent sequence in C, the limit is also in C.
   (a) Give an example of a closed set.
   (b) An open set O in $\mathbf{R}$ is a set whose complement (all the elements not in O) is closed. Show that if O is open, then for every a in O, it is possible to find $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset O$.

2.3 Series

A series, as we are going to see, is the sum of a sequence.

Definition
Let a_n be a sequence of real numbers. Define the sequence S_n by

$$S_n = \sum_{k=1}^{n} a_k.$$

S_n is the sequence of partial sums. The series $\sum_{k=1}^{\infty} a_k$ is said to converge if the sequence S_n converges. If that is the case, then the limit of S_n is denoted by $\sum_{k=1}^{\infty} a_k$.

The series above starts at $k = 1$, but it may actually start at any positive or 0 integer. A series that does not converge is said to diverge.

Example 2.14 Let r be a real number in $(-1, 1)$, and $a_n = r^n$. Recall that for any real numbers a, b and natural number n, we have

$$a^{n+1} - b^{n+1} = (a - b)\big(a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n\big).$$

Thus,

$$r^{n+1} - 1 = (r - 1)\big(r^n + r^{n-1} + \cdots + r + 1\big).$$

Since $r \neq 1$, we have

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}.$$

Defining $S_n = \sum_{k=0}^{n} a_k$, we get that

$$S_n = \frac{r^{n+1} - 1}{r - 1}.$$

By Example 2.3 in Sect. 2.1 we know that r^{n+1} converges to 0 as n goes to infinity for $|r| < 1$. Therefore,

$$\lim_{n\to\infty} S_n = \frac{1}{1 - r} \quad \text{for } |r| < 1.$$

In other words, the series $\sum_{k=0}^{\infty} a_k$ converges, and its sum is $\frac{1}{1-r}$.

The series considered in Example 2.14 is called a geometric series. It can be computed exactly, and as the reader will see, this is rather rare. We now turn to convergence issues. We start by a divergence test.

Divergence test
If the *series* $\sum_{k=1}^{\infty} a_k$ converges, then the *sequence* a_n converges to 0.

This test as indicated by its name can be used to show that a series diverges; it can *never* be used to show convergence.

If a_n does not converge to 0, the divergence test implies that the series $\sum_{k=1}^{\infty} a_k$ diverges. On the other hand, if the sequence a_n does converge to 0, then the test cannot be used: the series $\sum_{k=1}^{\infty} a_k$ may converge or may diverge. We will see examples of both situations.

We now prove the divergence test. Assume that the series $\sum_{k=1}^{\infty} a_k$ converges. By definition this means that $S_n = \sum_{k=1}^{n} a_k$ converges. Therefore, S_{n-1} converges to the same limit (why?), and so

$$a_n = S_n - S_{n-1} \quad \text{converges to 0.}$$

This completes the proof of the divergence test.

Example 2.15 Consider the geometric series $\sum_{k=0}^{\infty} r^k$ with $|r| \geq 1$. Then the sequence $a_n = |r^n| = |r|^n$ is bounded below by 1 and therefore cannot converge to 0. Thus, the series $\sum_{k=0}^{\infty} r^k$ diverges by the divergence test if $|r| \geq 1$.

Example 2.16 The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$. We are going to prove by induction that

$$S_{2^n} \geq \frac{n+2}{2}.$$

Let P denote the set of natural numbers n for which the inequality holds. Note that $S_2 = 1 + 1/2 = 3/2$ and that $\frac{1+2}{2}$ is also 3/2. So the inequality holds for $n = 1$. Assume that it holds for n. Then

$$S_{2^{n+1}} = S_{2^n} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geq S_{2^n} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}},$$

where the inequality comes from the fact that we replace all $1/k$ for k between 2^n and 2^{n+1} by the smallest of them, $1/2^{n+1}$. Note that

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} = \left(2^{n+1} - 2^n\right) \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Therefore, by the induction hypothesis,

$$S_{2^{n+1}} \geq S_{2^n} + \frac{1}{2} \geq \frac{n+2}{2} + \frac{1}{2} = \frac{n+3}{2}.$$

So $n+1$ belongs to P, and the inequality is proved.

The sequence S_{2^n} is larger than an unbounded sequence and therefore must be unbounded itself. Thus, S_{2^n} diverges. S_n has a subsequence that diverges and so cannot converge. Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This series gives an example for which $a_n = 1/n$ converges to 0 but the series $\sum_{n=1}^{\infty} a_n$ diverges. This provides a counterexample showing that the converse of the divergence test does not hold.

Operations on series
Assume that the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge. Let c be a constant. Then

(i) The series $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, and

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} a_k.$$

(ii) The series $\sum_{k=1}^{\infty}(ca_k)$ converges, and

$$\sum_{k=1}^{\infty}(ca_k) = c\sum_{k=1}^{\infty} a_k.$$

These properties are easily proved by applying the operations on limits of Sect. 2.2. Let

$$A_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad B_n = \sum_{k=1}^{n} b_k.$$

Since the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, we have that the sequences A_n and B_n converge. Hence, the sequence $A_n + B_n$ converges. Note that

$$A_n + B_n = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(a_k + b_k).$$

Hence, the convergence of the sequence $A_n + B_n$ is equivalent to the convergence of the series $\sum_{k=1}^{\infty}(a_k + b_k)$. Moreover,

$$\lim_{n\to\infty}(A_n + B_n) = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n.$$

That is,

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} a_k.$$

This proves (i). The proof of (ii) is similar and left to the reader.

We now turn to convergence tests for positive-term series. Positive-term series are easier to analyze because of the following obvious but important fact.

Positive-term series
Let a_n be a sequence, and $S_n = \sum_{k=1}^{n} a_k$ be the corresponding partial sums. The sequence S_n is increasing if and only if $a_n \geq 0$ for all $n \geq 1$.

Observe that

$$S_n - S_{n-1} = a_n,$$

and the result is obvious. As the reader will see, all the tests we will state in the rest of this section depend crucially on the properties of monotone sequences.

Comparison test
Consider two sequences of real numbers $a_n \geq 0$ and $b_n \geq 0$ such that for some natural N,

$$a_n \leq b_n \quad \text{for all } n \geq N.$$

(1) If the series $\sum_{n=0}^{\infty} b_n$ converges, so does the series $\sum_{n=0}^{\infty} a_n$.
(2) If the series $\sum_{n=0}^{\infty} a_n$ diverges, so does the series $\sum_{n=0}^{\infty} b_n$.

Let $S_n = \sum_{k=0}^{n} a_k$ and $T_n = \sum_{k=0}^{n} b_k$. We first prove (1). We assume that T_n converges. Take $n > N$. Then

$$S_n = \sum_{k=0}^{N} a_k + \sum_{k=N+1}^{n} a_k \leq S_N + \sum_{k=N+1}^{n} b_k.$$

Note that $\sum_{k=N+1}^{n} b_k = T_n - T_N$. Therefore,

$$S_n \leq S_N - T_N + T_n.$$

Since the series $\sum_{n=0}^{\infty} b_n$ converges, the sequence T_n is bounded and therefore bounded above by some K. Observe that since N is fixed, $S_N - T_N$ is a constant (that is, it does not depend on the variable n), and S_n is bounded above by $S_N - T_N + K$. On the other hand, $S_{n+1} - S_n = a_{n+1} \geq 0$, and thus S_n is an increasing sequence. An increasing sequence which is bounded above must converge, and (1) is proved.

Now we turn to (2). Assume that S_n diverges. We use the inequality proved above:

$$T_n \geq S_n - S_N + T_N.$$

As already observed, S_n is increasing; since it diverges, it cannot be bounded above. Thus, T_n is larger than $S_n - S_N + T_N$, which is not bounded above (why?), and so T_n cannot be bounded either. Therefore, T_n diverges, and (2) is proved.

Example 2.17 Let $a_n \geq 0$ and assume that the series $\sum_{n=1}^{\infty} a_n$ diverges. We show that $\sum_{n=1}^{\infty} \sqrt{a_n}$ diverges as well.

We consider two cases. If a_n does not converge to 0, then $\sqrt{a_n}$ does not converge to 0 either (why?), and by the divergence test the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ diverges.

If a_n converges to 0, there is N such that if $n \geq N$, then

$$|a_n - 0| < \epsilon = 1.$$

In particular, $0 \leq a_n < 1$ for $n \geq N$. Recall that $\sqrt{x} \geq x$ for x in [0, 1]. Hence,

$$\sqrt{a_n} \geq a_n \quad \text{for } n \geq N.$$

Since the series $\sum_{n=1}^{\infty} a_n$ diverges, so does the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ by the comparison test.

The following limit comparison test is sometimes easier to apply than the comparison test.

Limit comparison test
Consider two sequences of positive real numbers $a_n \geq 0$ and $b_n > 0$ such that $\frac{a_n}{b_n}$ converges to a strictly positive number. Then, the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge or both diverge.

We now prove the limit comparison test. Assume that a_n/b_n converges to some $\ell > 0$. By the definition of convergence with $\epsilon = \ell/2$, there is N such that if $n \geq N$,

$$\left| \frac{a_n}{b_n} - \ell \right| < \ell/2$$

or, equivalently,

$$\ell/2 < \frac{a_n}{b_n} < 3\ell/2 \quad \text{for } n \geq N.$$

Assume that the series $\sum_{n=0}^{\infty} a_n$ converges. Since

$$b_n < \frac{2}{\ell} a_n \quad \text{for } n \geq N,$$

the series $\sum_{n=0}^{\infty} b_n$ converges by the comparison test.

Assume now that the series $\sum_{n=0}^{\infty} a_n$ diverges. Since

$$b_n > \frac{2}{3\ell} a_n \quad \text{for } n \geq N,$$

the series $\sum_{n=0}^{\infty} b_n$ diverges by the comparison test.

To complete the proof, we should now show that if the series $\sum_{n=0}^{\infty} b_n$ converges, so does the series $\sum_{n=0}^{\infty} a_n$, and that if the series $\sum_{n=0}^{\infty} b_n$ diverges, so does the series $\sum_{n=0}^{\infty} a_n$. Instead, we may observe that b_n/a_n converges to $1/\ell$ (also a strictly positive real) and therefore a_n and b_n play symmetric roles here. This completes the proof of the limit comparison test.

Example 2.18 Does the series $\sum_{n=2}^{\infty} \frac{n}{n^2-1}$ converge?

Let $a_n = \frac{1}{n}$ and $b_n = \frac{n}{n^2-1}$. It is easy to check that a_n/b_n converges to 1. Since the series $\sum_{n=2}^{\infty} a_n$ diverges, so does the series $\sum_{n=2}^{\infty} b_n$ by the limit comparison test.

Example 2.19 Let $a \leq 1$ be a real number. The series

$$\sum_{k=1}^{\infty} \frac{1}{k^a}$$

diverges.

Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^a}$. Both sequences are positive, and since $a \leq 1$,

$$a_n \leq b_n \quad \text{for all } n \geq 1.$$

We have already shown that the series $\sum_{n=1}^{\infty} a_n$ diverges, so by the comparison test (with $N = 1$) the series $\sum_{n=1}^{\infty} b_n$ diverges as well.

We still need to analyze the case $a > 1$ for the series $\sum_{k=1}^{\infty} \frac{1}{k^a}$. In this case,

$$\frac{1}{k} > \frac{1}{k^a}.$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, this comparison is useless. We need another test.

Cauchy test
Consider a decreasing and positive sequence a_n. The series

$$\sum_{k=1}^{\infty} a_k$$

converges if and only if the series

$$\sum_{k=1}^{\infty} 2^k a_{2^k}$$

converges.

Let

$$S_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad T_n = \sum_{k=1}^{n} 2^k a_{2^k}$$

be the two partial sums.

Assume first that the series $\sum_{k=1}^{\infty} a_k$ converges and therefore that the sequence S_n converges. Consider

$$S_{2^k} - S_{2^{k-1}} = \sum_{j=2^{k-1}+1}^{2^k} a_j.$$

Since the sequence a_n is decreasing for every natural j in $[2^{k-1}+1, 2^k]$, we have $a_j \geq a_{2^k}$. Thus,

$$S_{2^k} - S_{2^{k-1}} \geq \sum_{j=2^{k-1}+1}^{2^k} a_{2^k} = 2^{k-1} a_{2^k},$$

where we use that in $[2^{k-1}+1, 2^k]$, there are 2^{k-1} naturals (why?). We sum the preceding inequality to get

$$\sum_{k=1}^{n} (S_{2^k} - S_{2^{k-1}}) \geq \sum_{k=1}^{n} 2^{k-1} a_{2^k} = \frac{1}{2} \sum_{k=1}^{n} 2^k a_{2^k} = T_n/2.$$

Observe that

$$\sum_{k=1}^{n} (S_{2^k} - S_{2^{k-1}}) = (S_2 - S_1) + (S_4 - S_2) + \cdots + S_{2^n} - S_{2^{n-1}} = S_{2^n} - S_1.$$

Hence,

$$S_{2^n} - S_1 \geq T_n/2.$$

That is,

$$T_n \leq 2S_{2^n} - 2S_1.$$

Since S_n converges, it must be bounded by some K. Thus,

$$T_n \leq 2K - 2S_1.$$

So T_n is bounded above. Since it is also increasing (why?), it must converge. We have proved the direct implication.

Assume now that T_n converges. Consider again

$$S_{2^k} - S_{2^{k-1}} = \sum_{j=2^{k-1}+1}^{2^k} a_j.$$

Since the sequence a_n is decreasing for every natural j in $[2^{k-1}+1, 2^k]$, we have $a_j \leq a_{2^{k-1}}$. Thus,

$$S_{2^k} - S_{2^{k-1}} \leq \sum_{j=2^{k-1}+1}^{2^k} a_{2^{k-1}} = 2^{k-1} a_{2^{k-1}}.$$

It follows that

$$\sum_{k=1}^{n} (S_{2^k} - S_{2^{k-1}}) \leq \sum_{k=1}^{n} 2^{k-1} a_{2^{k-1}} = T_{n-1} - a_1.$$

Therefore,

$$S_{2^n} - S_1 \leq T_{n-1} + a_1,$$

and since $S_1 = a_1$, we have

$$S_{2^n} \le T_{n-1} + 2a_1.$$

But $n < 2^n$ for every natural n. Using that S_n is an increasing sequence, we have

$$S_n \le S_{2^n}$$

for every n. Hence,

$$S_n \le S_{2^n} \le T_{n-1} + 2a_1.$$

Since T_n is increasing and convergent, it is bounded above, and so is S_n. We know that S_n is also increasing, so it converges. This proves the converse. The proof of the Cauchy test is complete.

Example 2.20 Let $p > 0$ be a real number and consider the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$. We have seen already that this series diverges for $p \le 1$. The Cauchy test will allow us to determine convergence in all cases. First, we need to check the hypotheses of the test. The sequence

$$a_k = \frac{1}{k^p}$$

is positive and decreasing for all $p > 0$. Note that

$$a_{2^k} = \frac{1}{(2^k)^p} = \left(2^{-p}\right)^k$$

and that the series of general term

$$2^k a_{2^k} = \left(2^{-p+1}\right)^k$$

is a geometric series with ratio $r = 2^{-p+1}$. In particular, $r < 1$ if and only if $p > 1$. By the Cauchy test, the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$.

We now have a complete result for series of the type $\sum_{k=1}^{\infty} \frac{1}{k^p}$ for real p. Since this is an important class of series, we summarize our results below.

p **Test**
The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges when $p > 1$ and diverges when $p \le 1$.

Exercises

1. Let $a_n = \frac{1}{n(n+1)}$ for $n \ge 1$.

(a) Compute $S_n = \sum_{k=1}^{n} a_k$ in function of n (observe that $a_n = 1/n - 1/(n+1)$).
(b) Show that the series $\sum_{k=1}^{\infty} a_k$ converges by computing its sum.

2. Does $\sum_{k=1}^{\infty}(-1)^k$ converge?
3. Compute $\sum_{n=1}^{\infty}(-1)^n \frac{3^n}{4^{n+1}}$.
4. Let $a_n \geq 0$ and $b_n = \sqrt{a_n}$. Show that a_n converges to 0 if and only b_n converges to 0.
5. (a) Let $a_n \geq 0$ be such that $\sum_{k=1}^{\infty} a_k$ converges. Show that $\sum_{k=1}^{\infty} a_k^2$ converges as well.
   (b) Is the converse of (a) true?
   (c) In (a) you have shown that if $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} f(a_k)$ converges for $f(x) = x^2$. Find other functions f for which this is true.
6. Let $a_n > 0$. Show that $\sum_{k=1}^{\infty} a_k$ converges if and only if

$$\sum_{k=1}^{\infty} \frac{a_k}{1+a_k}$$

   converges.
7. Show that $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=100}^{\infty} a_k$ converges.
8. Assume that the sequences $a_n \geq 0$ and $b_n > 0$ are such that a_n/b_n converges to 0.
   (a) Show that if the series $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.
   (b) Show that if the series $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$.
   (c) Show by exhibiting counterexamples that the converses of (a) and (b) do not hold.
9. The sequence a_n is said to go to $+\infty$ if for any real $A > 0$, there is a natural number N such that if $n \geq N$, then $a_n > A$.
   (a) Give an example of a sequence that goes to $+\infty$. Prove your claim.
   (b) Show that a sequence that goes to $+\infty$ cannot be bounded.
   (c) Is it true that any unbounded positive sequence goes to infinity?
10. Assume that the sequences $a_n \geq 0$ and $b_n > 0$ are such that a_n/b_n goes to infinity (see the definition in Exercise 9).
    (a) Show that if the series $\sum_{k=1}^{\infty} a_k$ converges, so does $\sum_{k=1}^{\infty} b_k$.
    (b) Show that if the series $\sum_{k=1}^{\infty} b_k$ diverges, so does $\sum_{k=1}^{\infty} a_k$.
    (c) Show by exhibiting counterexamples that the converses of (a) and (b) do not hold.
11. Assume that the sequences $a_n \geq 0$ and $b_n \geq 0$ are such that $\sum_{k=1}^{\infty} a_k$ converges and b_n is bounded. Prove that $\sum_{k=1}^{\infty} a_k b_k$ converges.
12. Assume that $a_n \geq 0$ and $b_n \geq 0$ for all $n \geq 1$. Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge. Prove that $\sum_{k=1}^{\infty} a_k b_k$ converges.
13. Prove that in $[2^{k-1}+1, 2^k]$ there are 2^{k-1} naturals.
14. (a) Show that if a_n converges, then $a_n - a_{n-1}$ converges to 0.
    (b) Is the converse of (a) true? Prove it or give a counterexample.

15. Assume that the series $\sum_{k=1}^{\infty} a_k$ converges. Let c be a constant. Show that the series $\sum_{k=1}^{\infty}(ca_k)$ converges and that

$$\sum_{k=1}^{\infty}(ca_k) = c\sum_{k=1}^{\infty} a_k.$$

16. Let d_n be a sequence in $\{0, 1, 2, \ldots, 9\}$. Show that the series

$$\sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

converges.

2.4 Absolute Convergence

We start with a definition.

Absolute convergence
The series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=0}^{\infty} |a_n|$ converges.

Example 2.21 Let $a_n = \frac{(-1)^n}{n^2}$. Then $|a_n| = \frac{1}{n^2}$. By the p rule, the series $\sum_{n=1}^{\infty} |a_n|$ converges. That is, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely. Does this imply that the series $\sum_{n=1}^{\infty} a_n$ also converges? As the following result shows, the answer to that question is yes.

Absolute convergence implies convergence
If a series converges absolutely, then it converges.

We now prove the result above. Assume that $\sum_{n=1}^{\infty} |a_n|$ converges. Note that for all reals x, we have

$$-|x| \le x \le |x|,$$

and so

$$0 \le x + |x| \le 2|x|.$$

Hence, for all naturals $n \ge 1$,

$$0 \le a_n + |a_n| \le 2|a_n|.$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} 2|a_n|$. By the comparison test (which is valid only for positive-term series), the series $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges as well.

Let $S_n = \sum_{k=1}^{n} a_k$ and $A_n = \sum_{k=1}^{n} |a_k|$. We have just shown that $S_n + A_n$ converges, and we know that A_n converges by hypothesis. Thus,

$$S_n = (S_n + A_n) - A_n$$

also converges (why?). This proves that the series $\sum_{n=1}^{\infty} a_n$ converges, and we are done.

Example 2.22 Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This series does not converge absolutely (why not?). However, as we will show below, it converges. Therefore, in general, convergence does not imply absolute convergence. Of course, if the a_n are all positive, then absolute convergence is the same as convergence.

We now turn to a useful test for absolute convergence.

Ratio Test
Assume that $a_n \neq 0$ for all n and that the sequence $|\frac{a_{n+1}}{a_n}|$ converges to some limit ℓ. If $\ell < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\ell > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. If $\ell = 1$, this test is inconclusive.

We prove the ratio test. Assume first that the limit ℓ exists and that $\ell < 1$. Let $\epsilon = (1 - \ell)/2 > 0$. By the definition of convergence, there is a natural N such that if $n \geq N$, then

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - \ell \right| < \epsilon.$$

Thus, for $n \geq N$,

$$|a_{n+1}| < |a_n|(\epsilon + \ell) = |a_n|(1 + \ell)/2.$$

Let $r = (1 + \ell)/2$ and note that $r < 1$. We have that

$$|a_{n+1}| < |a_n| r \quad \text{for all } n \geq N. \tag{2.2}$$

We will now show by induction that for all integers $k \geq 1$, we have

$$|a_{N+k}| < |a_N| r^k. \tag{2.3}$$

Let P be the set of naturals k for which (2.3) holds. Using (2.2) with $n = N$, we get

$$|a_{N+1}| < |a_N| r,$$

and 1 belongs to P. Assume now that natural k belongs to P. Using (2.2) with $n = N + k$, we have

$$|a_{N+k+1}| < |a_{N+k}| r.$$

By the induction hypothesis,

$$|a_{N+k}| < |a_N| r^k,$$

and so

$$|a_{N+k+1}| < |a_{N+k}|r < |a_N|r^k r = |a_N|r^{k+1}.$$

That is, $k+1$ belongs to P, and (2.3) is proved by induction.

Note that (2.3) can also be written as

$$|a_n| < |a_N|r^{n-N} \quad \text{for } n > N.$$

Observe also that since $r < 1$,

$$\sum_{n>N} |a_N|r^{n-N} = |a_N|\sum_{n\geq 1} r^n = |a_N|r\frac{1}{1-r}.$$

By the comparison test, the series $\sum_{n=1}^{\infty}|a_n|$ converges, and therefore the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. The ratio test is proved for $\ell < 1$.

We now turn to $\ell > 1$. Let $\epsilon = (\ell - 1)/2 > 0$. By the definition of convergence, there is a natural N such that if $n \geq N$, then

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \ell\right| < \epsilon.$$

Thus, for $n \geq N$,

$$|a_{n+1}| > |a_n|(-\epsilon + \ell) = |a_n|(1+\ell)/2 > a_n.$$

It is then easy to show, by induction, that

$$|a_n| > |a_N| \quad \text{for all } n > N.$$

This shows that the sequence a_n does not converge to 0 (why?). By the divergence test, the series $\sum_{n=1}^{\infty} a_n$ diverges.

The final claim to prove is that for $\ell = 1$, the test is not conclusive. In order to prove this claim, we need to find two examples with $\ell = 1$, one for which the series is convergent and one for which it is not. First, take $a_n = \frac{1}{n^2}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2$$

converges to $1^2 = 1$. That is, $\ell = 1$. By the p test we know that this series converges.

To get a divergent series, take $b_n = \frac{1}{n}$. Then

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{n}{n+1}$$

converges to 1. We have again $\ell = 1$, but this time the series diverges.

The ratio test is proved.

The ratio test is especially useful when the general term of the series a_n involves n as an exponent or as a factorial. We now give two such examples.

Example 2.23 Does $\sum_{n=1}(-1)^n \frac{3^n}{n^2}$ converge?

Let $a_n = (-1)^n \frac{3^n}{n^2}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = 3\frac{n^2}{(n+1)^2},$$

which converges to 3 as n goes to infinity. By the ratio test, the series diverges.

Example 2.24 Does the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converge?

Let $b_n = \frac{1}{n!}$. Then

$$\frac{|b_{n+1}|}{|b_n|} = \frac{n!}{(n+1)!} = \frac{1}{n+1},$$

which converges to 0 as n goes to infinity. By the ratio test, $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Example 2.25 The ratio test is really a comparison with a geometric series. This remark can be useful to compute numerical approximations. We now use this idea to compute a numerical approximation of the series $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$.

First, we check that this is a convergent series. Let

$$c_n = \frac{1}{n^2 2^n}.$$

Then

$$\frac{c_{n+1}}{c_n} = \frac{n^2}{(n+1)^2}\frac{2^n}{2^{n+1}} = \frac{1}{2}\frac{n^2}{(n+1)^2}$$

converges to $1/2$. By the ratio test, this series converges. We may use the first 10 terms (for instance) to get a numerical approximation of this series.

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n^2 2^n} = \frac{946538429}{1625702400} \sim 0.58.$$

The error is

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} - S_{10} \right| = \sum_{n=11}^{\infty} \frac{1}{n^2 2^n}.$$

However,

$$\frac{1}{n^2 2^n} \leq \frac{1}{11^2 2^n} \quad \text{for all } n \geq 1.$$

Hence,

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} - S_{10} \right| \leq \frac{1}{11^2} \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1}{11^2 2^{10}}.$$

Therefore, the approximation above has an error of at most $\frac{1}{2^{10}}$.

A test closely related to the ratio test is the following:

Root Test
Assume that the sequence $|a_n|^{1/n}$ converges to some limit ℓ. If $\ell < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\ell > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. If $\ell = 1$, this test is inconclusive.

Assume first that $|a_n|^{1/n}$ converges to some limit $\ell < 1$. Let $r = \frac{1+\ell}{2} < 1$. Take $\epsilon = \frac{1-\ell}{2} > 0$. There is N such that if $n \geq N$, then

$$\left||a_n|^{1/n} - \ell\right| < \epsilon = \frac{1-\ell}{2}.$$

Hence,

$$|a_n|^{1/n} < \frac{1+\ell}{2} = r \quad \text{for all } n \geq N$$

and

$$|a_n| < r^n \quad \text{for all } n \geq N.$$

Since the geometric series $\sum_{n=1}^{\infty} r^n$ converges ($r < 1$), so does the series $\sum_{n=1}^{\infty} a_n$. This proves the first half of the test.

For the second half, assume that $|a_n|^{1/n}$ converges to some limit $\ell > 1$. Take $\epsilon = \frac{\ell-1}{2} > 0$. There is N such that if $n \geq N$, then

$$\left||a_n|^{1/n} - \ell\right| < \epsilon = \frac{\ell-1}{2}.$$

Hence,

$$|a_n|^{1/n} > \frac{\ell+1}{2} \quad \text{for all } n \geq N$$

and

$$|a_n| > \left(\frac{\ell+1}{2}\right)^n \quad \text{for all } n \geq N.$$

Since $\frac{\ell+1}{2} > 1$, $|a_n|$ does not converge to 0, and by the divergence test the series $\sum_{n=1}^{\infty} a_n$ diverges. This completes the proof of the root test.

The ratio test is usually easier to use than the root test. However, the root test is useful in series having terms n^n such as in the following example.

Example 2.26 Consider the convergence and numerical approximation of the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$.

Let $a_n = \frac{1}{n^n}$. Then

$$|a_n|^{1/n} = \frac{1}{n}$$

converges to 0. By the root test, this series converges absolutely. We now find a numerical approximation. We have

$$S_4 = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} = \frac{8923}{6912}.$$

The error is

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^n} - S_4 \right| = \sum_{n=5}^{\infty} \frac{1}{n^n}.$$

For $n \geq 5$, we have

$$n^n \geq 5^n,$$

and so

$$\frac{1}{n^n} \leq \frac{1}{5^n}.$$

Hence, by taking only the first four terms the error is less than

$$\sum_{n=5}^{\infty} \frac{1}{5^n} = \frac{1}{5^5} \frac{1}{1-1/5} = \frac{1}{2500}.$$

We now turn to alternating series.

Alternating Series Theorem
Assume that $a_n \geq 0$ for all $n \geq 1$, a_n is a decreasing sequence, and that a_n converges to 0. Then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

Let $S_n = \sum_{k=1}^{n} (-1)^k a_k$ and $E_n = S_{2n}$. We have that

$$\begin{aligned} E_{n+1} - E_n = S_{2n+2} - S_{2n} &= (-1)^{2n+2} a_{2n+2} + (-1)^{2n+1} a_{2n+1} \\ &= a_{2n+2} - a_{2n+1} \leq 0, \end{aligned}$$

where the inequality comes from the assumption that a_n is a decreasing sequence. Since $E_{n+1} - E_n \leq 0$, E_n is also decreasing. We now show that it is bounded below. Note that

$$E_1 = -a_1 + a_2 \geq -a_1.$$

Similarly,

$$E_2 = -a_1 + (a_2 - a_3) + a_4 \geq -a_1$$

since $a_2 - a_3 \geq 0$ and $a_4 \geq 0$. More generally, taking $n \geq 3$, we have that

$$E_n = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{2n-2} - a_{2n-1}) + a_{2n} \geq -a_1.$$

That is, the sequence E_n is decreasing and bounded below by $-a_1$; therefore, it converges to some limit ℓ. Let $F_n = S_{2n-1}$ for $n \geq 1$. Then

$$E_n - F_n = a_{2n} \geq 0.$$

Since a_n converges to 0, so does a_{2n} (why?), and by the squeezing principle, $E_n - F_n$ converges to 0 as well. Therefore, F_n converges to ℓ. At this point we have that S_{2n} and S_{2n+1} converge to the same limit ℓ. We now show that this implies that the whole sequence S_n converges to ℓ. Take any $\epsilon > 0$. There are N_1 and N_2 such that if $n \geq N_1$, then

$$|S_{2n} - \ell| < \epsilon,$$

and if $n \geq N_2$, then

$$|S_{2n-1} - \ell| < \epsilon.$$

Let $N = \max(N_1, N_2)$ and take $n \geq 2N$. Either n is even and $n = 2k$ with $k \geq N \geq N_1$, so

$$|S_{2k} - \ell| < \epsilon,$$

or n is odd and $n = 2k - 1$ with $k \geq N \geq N_2$, so

$$|S_{2k-1} - \ell| < \epsilon.$$

We have shown that for any $\epsilon > 0$, there is $2N$ such that if $n \geq 2N$, then

$$|S_n - \ell| < \epsilon.$$

That is, the alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

Numerical approximations are easy to compute for alternating series.

Numerical approximations

Assume that $a_n \geq 0$ for all $n \geq 1$, a_n is a decreasing sequence, and that a_n converges to 0. Then, for any natural k, we have

$$\sum_{n=1}^{2k-1} (-1)^n a_n \leq \sum_{n=1}^{\infty} (-1)^n a_n \leq \sum_{n=1}^{2k} (-1)^n a_n.$$

That is, *if* the alternating series theorem applies, then a partial sum gives an approximation whose error is less than the first omitted term.

In order to prove the double inequality above, we use the notation and the results above. In particular, we have shown that $E_n = S_{2n}$ is decreasing and convergent. Therefore, the limit ℓ of E_n (which is also the infinite series) is less than E_n for every $n \geq 1$. This proves the upper bound. For the lower bound, consider $F_n = S_{2n-1}$. We have

$$F_{n+1} - F_n = -a_{2n+1} + a_{2n} \geq 0$$

since a_n is decreasing. That is, F_n is increasing. We also know that F_n converges to the same ℓ. Therefore, ℓ is larger than F_n for all $n \geq 1$. This provides the lower bound above, and we are done.

Example 2.27 Consider $\sum_{n=1}^{\infty}(-1)^n a_n$ with $a_n = 1/n$. Note that $a_n > 0$ for all n, a_n is decreasing, and that it converges to 0. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. Moreover, we have

$$S_1 = -1, \qquad S_2 = -1 + 1/2 = -1/2, \qquad S_3 = -1/2 - 1/3 = -5/6,$$
$$S_4 = -5/6 + 1/4 = -7/12.$$

In particular,

$$-5/6 \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \leq -7/12.$$

Exercises

1. Show that for every real x, we have

$$0 \leq x + |x| \leq 2|x|.$$

2. Assume that the sequences $a_n + b_n$ and a_n converge. Show that the sequence b_n converges as well.
3. Assume that $a_n \neq 0$ for all n.
   (a) Show that if $|a_{n+1}| > |a_n|$ for every $n \geq N$, then $|a_n| > |a_N|$ for all $n > N$.
   (b) Show that if the conclusion of (a) holds, then the sequence a_n does not converge to 0.
4. Does the series $\sum_{n=1}(-1)^n \frac{n}{2^n}$ converge?
5. (a) Does the series $\sum_{n=1} \frac{n!}{(2n)!}$ converge?
   (b) Use (a) to find the limit of the sequence $\frac{n!}{(2n)!}$.
6. Does the series $\sum_{n=1} \frac{n!}{2^n}$ converge?
7. Let $a_n = 1/3^n$ if n is not a prime and $a_n = n^2/3^n$ if n is a prime. Does the series $\sum_{n=1} a_n$ converge?
8. Under the assumptions of the alternating series theorem, show that

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

   both converge.

9. Find a numerical approximation for the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.
10. Assume that $|a_n|^{1/n}$ converges to some ℓ.
    (a) Show that if $\ell = 1$, then the root test is not conclusive (you may use that $n^{1/n}$ converges to 1).
    (b) Show that if $\ell > 1$, then there is $r > 1$ and a natural N such that if $n \geq N$, then
$$|a_n| \geq r^n.$$
11. (a) Assume that a_{3n}, a_{3n+1}, a_{3n+2} all converge to the same limit ℓ. Show that the sequence a_n converges.
    (b) Generalize the result in (a).
12. Assume that $a_n \geq 0$ for every n and that the series $\sum_{n=1}^{\infty} a_n$ converges.
    (a) Show that $\sum_{n=1}^{\infty} (-1)^n a_n$ converges as well.
    (b) Does (a) hold if the sequence a_n is not assumed to be positive?
13. Assume that $|\frac{a_{n+1}}{a_n}|$ converges to some limit $\ell > 1$.
    (a) Show that there exist a real $r > 1$ and a natural N such that for $n > N$, we have that
$$|a_n| > r^{n-N} |a_N|.$$
    (b) Show that $|a_n|$ goes to infinity as n goes to infinity. That is, show that for every A, there is M such that if $n \geq M$, then $|a_n| > A$.
14. How many terms do we need to take to have a numerical approximation of $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$ with an error less than 10^{-10}? Prove your claim.
15. (a) Find an upper bound for
$$\sum_{n=k+1}^{\infty} \frac{1}{n^n}$$
as a function of k.
    (b) Find a bound on the error when $\sum_{n=1}^{\infty} \frac{1}{n^n}$ is approximated by $\sum_{n=1}^{10} \frac{1}{n^n}$.
16. In the exercises of Sect. 2.3 it was shown that $a_n \geq 0$ for every n and that if the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges as well. Does this hold if the sequence is not assumed to be positive?
17. (a) Assume that the series $\sum_{k=1}^{\infty} |a_k - a_{k-1}|$ converges. Show that the sequence a_n converges.
    (b) Suppose that for every $k \geq 1$, we have
$$|a_k - a_{k-1}| \leq \frac{1}{2^k}.$$
Show that the sequence a_n converges.

Chapter 3
Power Series and Special Functions

3.1 Power Series

A power series is a function defined as a series.

Power series
A power series is a function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where c_n is a sequence of real numbers.

Example 3.1 The simplest example of power series is given by the geometric series. Recall from Example 2.10 in Sect. 2.2 that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for all } |x| < 1$$

and that the series diverges for $|x| \geq 1$. Therefore, the power series

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is defined for x in $(-1, 1)$. Note that the power series above is obtained when $c_n = 1$ for all $n \geq 0$.

The first task when studying a power series is to decide where it is defined. The following result shows that the domain of a power series is always an interval.

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Radius of convergence
A power series $\sum_{n=0}^{\infty} c_n x^n$ has a *radius of convergence* R. The power series converges if $|x| < R$, diverges if $|x| > R$, and can go either way if $|x| = R$. We may have $R = +\infty$, in which case the power series converges for all x. Moreover, R is the least upper bound (if it exists!) of the set

$$I = \{r \geq 0 : \text{ the sequence } c_n r^n \text{ is bounded}\}.$$

In particular, if $|x| > R$, then the sequence $c_n x^n$ is not bounded.

To prove the result above, we first need a lemma due to Abel.

Abel's Lemma *Assume that the sequence $c_n b^n$ is bounded for a real $b \neq 0$. Then the power series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for any x such that $|x| < |b|$.*

We now prove Abel's lemma. There is a real A such that for all $n \geq 1$,

$$\left|c_n b^n\right| < A.$$

Take now x such that $|x| < |b|$. For $n \geq 1$, we have

$$\left|c_n x^n\right| = \left|c_n b^n\right| \left|\frac{x}{b}\right|^n < A \left|\frac{x}{b}\right|^n.$$

The series $\sum_{n=0}^{\infty} |\frac{x}{b}|^n$ converges since it is a geometric series with $r = |\frac{x}{b}| < 1$. By the comparison test the (positive terms), series $\sum_{n=0}^{\infty} |c_n x^n|$ converges as well. This proves Abel's lemma.

We now turn to the proof of the existence of the radius of convergence R. Let

$$I = \{r \geq 0 : \text{ the sequence } c_n r^n \text{ is bounded}\}.$$

Note that the sequence $c_n r^n$ is bounded by $|c_0|$ when $r = 0$. Thus, 0 belongs to I, and I is not empty. There are two cases:

1. If I is bounded above, then, by the fundamental property of the reals, it has a least upper bound $R \geq 0$. If $R = 0$, then $c_n x^n$ is not bounded for any $x \neq 0$. By the divergence test, the power series converges for $x = 0$ only. Assume now that $R > 0$. For any x such that $|x| < R$, there must be at least one r in I such that $|x| < r < R$ (if all r in I are below $|x|$, then $|x|$ is an upper bound of I smaller than R, and that is not possible). By Abel's lemma the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely. This is true for any $|x| < R$. On the other hand, if $|x| > R$, then $|x|$ does not belong to I, and therefore $c_n |x|^n$ is not bounded. So the power series cannot converge at x for $|x| > R$.
2. If I is not bounded above, for any x, there is r such that r is in I and $|x| < r$. By Abel's lemma the power series converges absolutely for x. Thus, the power series converges absolutely for all x. We set $R = +\infty$. This completes the description of the domain of a power series.

The next example illustrates the fact that the ratio test is many times useful to find the radius of convergence of a series. It also shows that convergence at end points can go either way.

Example 3.2 What is the domain of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n} x^n$?

Let $a_n = \frac{(-1)^n}{n2^n} x^n$. Then, $\frac{|a_{n+1}|}{|a_n|} = \frac{n}{n+1}\frac{1}{2}|x|$ converges to $|x|/2$. By the ratio test, the power series converges if $|x|/2 < 1$ and diverges if $|x|/2 > 1$. Thus, the radius of convergence for this series is 2. At this point we know that the domain of the series includes $(-2, 2)$, but it may or may not include one or both of the end points of this interval.

We now examine the endpoints. Plugging $x = 2$ into the series gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the alternating series theorem. On the other hand, setting $x = -2$ in the power series gives $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (why?). Therefore, the domain of this power series is $(-2, 2]$.

Exercises

1. What is the radius of convergence of the series $\sum_{n=0}^{\infty} n!x^n$?
2. What is the domain of the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}3^n} x^n$?
3. What is the domain of the series $\sum_{n=0}^{\infty} \frac{n}{n+1} x^n$?
4. Assume that $|c_{n+1}|/|c_n|$ converges to some limit ℓ.
   (a) Show that if $\ell = 0$, then the radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$ is infinite.
   (b) Show that if $\ell > 0$, then the radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$ is $1/\ell$.
5. Assume that $|c_{n+1}|/|c_n|$ goes to infinity. That is, assume that for any $A > 0$, there is a natural N such that if $n \geq N$, then
$$|c_{n+1}|/|c_n| > A.$$
   Show that the power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is defined at $x = 0$ only.

3.2 Trigonometric Functions

We start by defining the functions sine and cosine using power series.

Sine and Cosine as power series
We *define*, for every real x,

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

For the definition above to make sense, the two power series must converge for every x. We now check that. Let

$$a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Then

$$a_{n+1} = (-1)^{n+1} \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} = (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!}$$

and

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2n+1)!}{(2n+3)!} \frac{|x|^{2n+3}}{|x|^{2n+1}} = \frac{1}{(2n+2)(2n+3)} |x|^2.$$

As n goes to infinity, $\frac{|a_{n+1}|}{|a_n|}$ converges to $0 < 1$. By the ratio test the power series converges absolutely for every x. Thus, the function sin is defined everywhere. The proof for cos is very similar and is left as an exercise for the reader.

First properties
We have $\cos 0 = 1$ and $\sin 0 = 0$. The function cos is even, and the function sin is odd.

The power series definition gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Letting $x = 0$ gives $\sin 0 = 0$ and $\cos 0 = 1$. Since sin has only odd powers, we have $\sin(-x) = -\sin x$. On the other hand, cos has only even powers, so $\cos(-x) = \cos x$. This proves the first properties.

If we differentiate term by term the power series defining $\sin x$, we get

$$(\sin x)' = 1 - \frac{3x^2}{3!} + \frac{5x^5}{5!} - \frac{7x^7}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Thus, it *seems* that the derivative of sin is cos. This is, of course, true. However, in order to *prove* this, one needs to be able to differentiate a power series term by term. Since there are infinitely many terms in a series, this is not at all trivial. It will be shown in Chap. 7 that a power series with radius of convergence R is infinitely differentiable on $(-R, R)$ and that one can differentiate it term by term. Moreover, the differentiated power series all have the same radius of convergence as the initial power series.

We now do the same type of informal computation for cos, and we get

$$(\cos x)' = -\frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \cdots = -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right).$$

Again, this seems to indicate that $(\cos x)' = -\sin x$, and this is true.

Note that

$$(\sin x)'' = \big((\sin x)'\big)' = (\cos x)' = -\sin x.$$

Likewise,

$$(\cos x)'' = -\cos x.$$

That is, sin and cos are both solutions of the differential equation $f'' + f = 0$. The next lemma will help us show that this differential equation characterizes sin and cos.

Lemma *Suppose that f is a function twice differentiable on* $\mathbf{R}$, *that*

$$f'' + f = 0,$$

and that $f(0) = f'(0) = 0$. Then is f is identically 0.

Define $h = f^2 + (f')^2$. Since f is differentiable, $f^2 = f \cdot f$ is also differentiable. The function f' is differentiable, so $(f')^2$ is differentiable as well. Applying the chain rule from Calculus gives

$$h' = 2ff' + 2f'f'' = 2f'(f + f'') = 0,$$

where we use the hypothesis $f'' + f = 0$. A differentiable function whose derivative is 0 on an interval is a constant on that interval. Thus, h is a constant. Since $h(0) = 0$, the function h is identically 0. Hence, $h = f^2 + (f')^2$ is identically 0. Assume now, by contradiction, that there is a real a such that $f(a) \neq 0$. Then

$$h(a) = f(a)^2 + \big(f'(a)\big)^2 \geq f(a)^2 > 0.$$

Therefore, h is not identically 0. We have a contradiction. For all reals a, $f(a) = 0$. This completes the proof of the lemma.

Sine and Cosine as solutions of a differential equation

The function sin is the only solution of the differential equation

$$f'' + f = 0 \quad \text{with } f(0) = 0 \text{ and } f'(0) = 1.$$

The function cos is the only solution of the differential equation

$$f'' + f = 0 \quad \text{with } f(0) = 1 \text{ and } f'(0) = 0.$$

We now prove the claim above. Assume that there are two functions f and g such that

$$f'' + f = 0 \quad \text{with } f(0) = 0 \text{ and } f'(0) = 1,$$
$$g'' + g = 0 \quad \text{with } g(0) = 0 \text{ and } g'(0) = 1.$$

Subtracting the two equations, we get

$$f'' - g'' + f - g = 0.$$

Let $h = f - g$. Then $h'' = f'' - g''$, and we have

$$h'' + h = 0.$$

Moreover, $h(0) = f(0) - g(0) = 0$ and $h'(0) = f'(0) - g'(0) = 1 - 1 = 0$. Therefore, according to the lemma, h is identically 0. That is, f and g are the same function. In other words, sin is the only function such that $\sin 0 = 0$, $(\sin)'(0) = \cos 0 = 1$ and $(\sin)'' = -\sin$. We leave the cos result as an exercise for the reader.

The characterization of sin and cos as solutions of differential equations turn out to be quite useful to prove many properties of cos and sin. We start with

Properties of Sine and Cosine

(i) $\sin^2 x + \cos^2 x = 1$ for all real x.
(ii) $|\sin x| \le 1$ and $|\cos x| \le 1$.
(iii) $\sin(x + y) = \sin x \cos y + \sin y \cos x$ for all reals x, y.
(iv) $\cos(x + y) = \cos x \cos y - \sin y \sin x$ for all reals x, y.

We start by proving (i). Let $h(x) = \sin^2 x + \cos^2 x$. Since sin and cos are differentiable, so is h. By the chain rule,

$$h'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0.$$

Thus, h is constant on the reals. Since $h(0) = 1$, (i) is proved.

Note that

$$\sin^2 x \le \sin^2 x + \cos^2 x = 1.$$

Thus,

$$\sin^2 x \le 1, \quad \text{and therefore,} \quad |\sin x| \le 1.$$

The same argument shows that $|\cos x| \le 1$. This proves (ii).

We now prove (iii). Let a be a fixed real, and let g be the function

$$g(x) = \sin(a + x) - \sin x \cos a - \sin a \cos x.$$

The function g is differentiable, and its derivative is

$$g'(x) = \cos(a + x) - \cos x \cos a + \sin a \sin x.$$

We differentiate g' to get, for every real x,

$$g''(x) = -\sin(a + x) + \sin x \cos a + \sin a \cos x = -g(x).$$

That is, g is a solution of the differential equation $f'' + f = 0$. Moreover, $g(0) = 0$ (why?) and $g'(0) = 0$. According to the lemma above, g is identically 0, and (iii) is proved. Statement (iv) can be proved in a similar way, and we leave the proof to the reader.

If we let $x = y$ in (iii) and (iv), we get the following useful formulas.

Double angles formulas

$$\sin(2x) = 2\sin x \cos x$$

and

$$\cos(2x) = \cos^2 x - \sin^2 x.$$

We now define the number π.

The number π
The equation $\cos x = 0$ has a smallest positive solution. This solution is denoted by $\pi/2$.

The proof of this result depends critically on the continuity of cos and sin. We quickly recall the definition; a more in depth study of continuity will be done in Chap. 5.

Continuity
Suppose that the function f is defined on a set D. The function f is said to be continuous at a in D if for every sequence a_n in D that converges to a, we have that $f(a_n)$ converges to $f(a)$.

We will need two steps in order to prove that $\cos x = 0$ has a smallest positive solution.

In the first step we show that there is at least one positive solution of the equation $\cos x = 0$. Recall, from Calculus the mean value theorem: if f is differentiable on (a, b) and continuous on $[a, b]$, there is c in (a, b) such that

$$f(b) - f(a) = (b - a)f'(c).$$

We apply the mean value theorem to the function sin. There exists a number c in (0,2) such that

$$\sin 2 - \sin 0 = 2\cos c.$$

Hence,

$$2\cos c = \sin 2.$$

In particular,

$$|\cos c| = |\sin 2|/2 \leq 1/2.$$

Note that if $|a| \leq 1/2$, then $a^2 \leq 1/4$ (why?). Hence,

$$\cos^2 c < 1/4. \tag{3.1}$$

We now show that $\cos(2c) < 0$. We have

$$\cos(2c) = \cos^2 c - \sin^2 c = \cos^2 c - \left(1 - \cos^2 c\right) = 2\cos^2 c - 1.$$

Using (3.1), we have

$$2\cos^2 c - 1 \leq 2(1/4) - 1 < 0.$$

Thus, the function cos is strictly negative at $2c$. But $\cos 0 = 1$. Since cos is a continuous function, in order to go from a positive value to a negative value, it must go through 0. More mathematically, the intermediate value theorem states that there is d in $(0, 2c)$ such that $\cos d = 0$. This completes our first step.

In the second step we show that the set of positive solutions of the equation $\cos x = 0$ has a smallest element. Let

$$A = \{x \geq 0 : \cos x = 0\}.$$

A is bounded below by 0 and is nonempty (by the first step), so it has a greatest lower bound m. We know that there is a sequence a_n in A that converges to m. By our definition of continuity,

$$\lim_{n\to\infty} \cos a_n = \cos m$$

since cos is continuous everywhere. But $\cos a_n$ is 0 for every $n \geq 0$ (why?), so $\cos m = 0$ as well. That is, the greatest lower bound of A is in A. No positive solution of $\cos x = 0$ can be smaller than m (by the definition of m). Moreover, m is a solution. Thus, we have found a number m which is the smallest positive solution of $\cos x = 0$.

This last step is a little subtle: it could be the case that the set of solutions is $\{1/n, n \geq 1\}$. In that case there is no smallest solution (why?). We have ruled out this possibility for the function cos.

We define $\pi = 2m$. This completes the second step and our definition of π.

We now list a few important formulas involving π.

More trig

We have

(i) $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$.
(ii) $\cos(\pi/2 - x) = \sin x$ and $\sin(\pi/2 - x) = \cos x$.
(iii) $\cos(\pi/2 + x) = -\sin x$ and $\sin(\pi/2 + x) = \cos x$.

The fact that $\cos(\pi/2) = 0$ comes from the definition of $\pi/2$. We now compute $\sin(\pi/2)$. Since $m = \pi/2$ is the smallest positive solution of $\cos x = 0$ and $\cos 0 = 1$, we have that $\cos x > 0$ for x in $(0, \pi/2)$ (why?). Since $(\sin)' = \cos$, this tells us that sin is increasing on $(0, \pi/2)$. On the other hand,

$$\cos^2(\pi/2) + \sin^2(\pi/2) = 1.$$

Thus, $\sin(\pi/2) = 1$ or -1. But $\sin 0 = 0$, and sin is increasing on $[0, \pi/2]$, so $\sin(\pi/2) = 1$. This proves (i).

We now prove (iii). (ii) is very similar and left to the reader. Use the addition formulas to get

$$\sin(x + \pi/2) = \sin(x)\cos(\pi/2) + \cos(x)\sin(\pi/2) = \cos(x)$$

and

$$\cos(x + \pi/2) = \cos(x)\cos(\pi/2) - \sin(x)\sin(\pi/2) = -\sin(x).$$

This proves (iii).

A function f is said to have a period p if $f(x + p) = f(x)$ for every x.

Sine and Cosine are periodic
The functions sin and cos have period 2π.

We prove that sin has period 2π. The proof for cos is similar and left to the reader. By (iii),

$$\sin(x + \pi) = \sin(x + \pi/2 + \pi/2) = \cos(x + \pi/2) = -\sin(x).$$

Hence,

$$\sin(x + 2\pi) = \sin(x + \pi + \pi) = -\sin(x + \pi) = \sin(x).$$

This proves that sin has period 2π.

It is useful to know sin and cos for some remarkable reals.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1

We have already computed the values for 0 and $\pi/2$. We now compute the values for π:

$$\sin(\pi) = \sin(2\pi/2) = 2\sin(\pi/2)\cos(\pi/2) = 0,$$
$$\cos(\pi) = \cos(2\pi/2) = \cos^2(\pi/2) - \sin^2(\pi/2) = -1.$$

For $\pi/4$, we remark that

$$\cos(\pi/2) = \cos(2\pi/4) = \cos^2(\pi/4) - \sin^2(\pi/4).$$

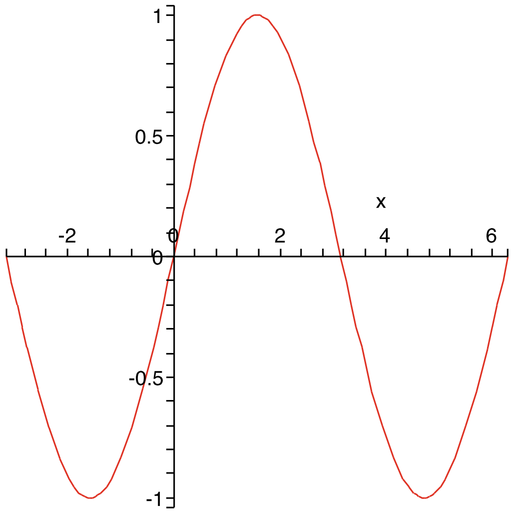

Fig. 3.1 This is the graph of the sine function

Hence,

$$\cos^2(\pi/4) = \sin^2(\pi/4).$$

Since

$$\cos^2(\pi/4) + \sin^2(\pi/4) = 1,$$

we have that

$$\cos^2(\pi/4) = \sin^2(\pi/4) = 1/2.$$

But cos is strictly positive on $(0, \pi/2)$; therefore,

$$\cos(\pi/4) = \sqrt{2}/2.$$

As noted above, since cos is the derivative of sin, sin is increasing on $(0, \pi/2)$. Since $\sin 0 = 0$, we have

$$\sin(\pi/4) = \sqrt{2}/2.$$

The computations for $\pi/3$ and $\pi/6$ are left as exercises.

Application 3.1 Sketch the graphs of sin and cos.

Figure 3.1 is the graph of sin. We now indicate how to sketch this graph. As observed above, sin is increasing on $(0, \pi/2)$, $\sin 0 = 0$, and $\sin(\pi/2) = 1$. By (ii),

$$\sin(\pi/2 - x) = \cos(x),$$

and by (iii),

$$\sin(\pi/2 + x) = \cos(x).$$

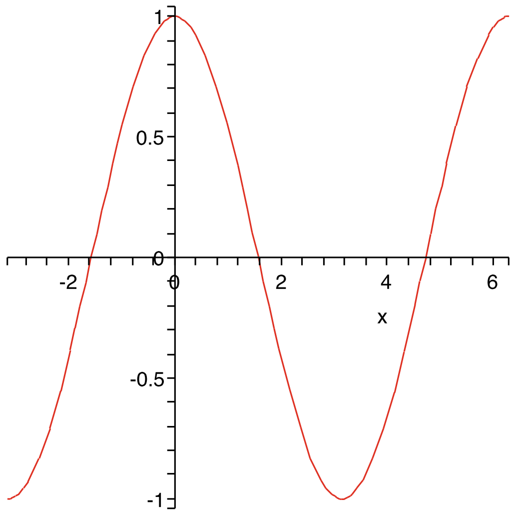

Fig. 3.2 This is the graph of the cosine function

Hence, $\sin(\pi/2 - x) = \sin(\pi/2 + x)$. That is, the graph of sin is symmetric with respect to the line $x = \pi/2$. In particular, since sin is increasing on $(0, \pi/2)$, it must be decreasing on $(\pi/2, \pi)$. The second derivative of sin is $-\sin$ which is always negative on $(0, \pi)$. Thus, sin is concave down on $(0, \pi)$. Since sin is odd, its graph is symmetric with respect to the origin. With these remarks, we can sketch the graph of sin on $(-\pi, \pi)$ (which is a period), and it can be completed by using the 2π periodicity of sin.

Once we have the graph of sin, we have the graph of cos since

$$\cos(x) = \sin(x + \pi/2).$$

This shows that it is enough to shift the graph of sin by $-\pi/2$ along the x axis to get the graph of cos see Fig. 3.2.

We now go back to the power series expressions to compute numerically $\sin x$ and $\cos x$. The following are useful bounds for sin and cos.

Bounds for Sine and Cosine

For every real $x \geq 0$ and integer $k \geq 1$, we have

$$1 - \frac{x^2}{2} + \cdots - \frac{x^{4k-2}}{(4k-2)!} \leq \cos x \leq 1 - \frac{x^2}{2} + \cdots - \frac{x^{4k-2}}{(4k-2)!} + \frac{x^{4k}}{(4k)!},$$

$$x - \frac{x^3}{3!} + \cdots - \frac{x^{4k-1}}{(4k-1)!} \leq \sin x \leq x - \frac{x^3}{3!} + \cdots - \frac{x^{4k-1}}{(4k-1)!} + \frac{x^{4k+1}}{(4k+1)!}.$$

In words, the bounds above tell us that if we use the first n terms of the power series expansion for $\sin x$ or $\cos x$, then the error we make (to estimate the infinite series) is less than the $(n+1)$th term. If the finite sum ends with a positive term, we are overestimating, while if the finite sum ends with a negative term, we are underestimating.

We prove the bounds above for some particular cases. We will use repeatedly the following property from Calculus. Assume that $a < b$ and that f and g are continuous functions on $[a, b]$ such that

$$f(x) \leq g(x) \quad \text{for all } x \in [a, b].$$

Then,

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

We start with

$$\cos u \leq 1.$$

We integrate the inequality above between 0 and $x > 0$:

$$\int_0^x \cos u\,du \leq \int_0^x 1\,du.$$

By the fundamental theorem of Calculus and since sin is an antiderivative of cos, we get

$$\sin x - \sin 0 \leq x - 0.$$

That is,

$$\sin x \leq x \quad \text{for all } x \geq 0.$$

We integrate the preceding inequality to get

$$\int_0^x \sin u\,du \leq \int_0^x u\,du,$$
$$-\cos x + 1 \leq x^2/2 - 0.$$

That is,

$$\cos x \geq 1 - x^2/2.$$

By integrating again we get

$$\sin x \geq x - x^3/3!.$$

Integrating twice gives

$$\cos x \leq 1 - x^2/2! + x^4/4!$$

and

$$\sin x \leq x - x^3/3! + x^5/5!.$$

So at this point we have that

$$1 - x^2/2 \leq \cos x \leq 1 - x^2/2 + x^4/4!$$

and

$$x - x^3/3! \leq \sin x \leq 1 - x^3/3! + x^5/5!.$$

The formulas above have been proved for $k = 1$. In order to establish the general formula, one may use induction, and this is left as an exercise.

Example 3.3 Compute an approximate value for $\sin 2$. We compute the first terms in the series for $x = 2$. We have $2^3/3! = 8/6 = 4/3$, $2^5/5! = 32/120 = 4/15$, $2^7/7! = 8/315$, and $2^9/9! = 4/2835$. Thus,

$$2 - 2^3/3! + 2^5/5! - 2^7/7! \leq \sin 2 \leq 2 - 2^3/3! + 2^5/5! - 2^7/7! + 2^9/9!.$$

That is,

$$286/315 \leq \sin 2 \leq 2578/2835.$$

By taking the midpoint of the interval as an approximation for $\sin 2$ we make an error less than $\frac{1}{2}\frac{2^9}{9!} = 2/2835$.

Example 3.4 Does the series $\sum_{n=1}^{\infty} \sin(1/n)$ converge?

Note that

$$\frac{1}{n} - \frac{1}{3!n^3} \leq \sin(1/n) \leq \frac{1}{n}.$$

Hence,

$$1 - \frac{1}{3!n^2} \leq \frac{\sin(1/n)}{1/n} \leq 1.$$

It is easy to see that for every $n \geq 1$, we have $1 - \frac{1}{3!n^2} > 0$. Hence, by the squeezing principle,

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1.$$

Since $\sum_{n=1}^{\infty} \sin(1/n)$ is a positive-term series, the limit comparison test applies. By the p test the series $\sum_{n=1}^{\infty} 1/n$ diverges, and so does $\sum_{n=1}^{\infty} \sin(1/n)$.

Exercises

1. Show that the function cos is defined everywhere.
2. Show that the set of solutions of $\cos x = 0$ is $\{\pi/2 + k\pi; k \in \mathbf{Z}\}$.
3. Prove that the function cos is the only solution of the differential equation

$$f'' + f = 0 \quad \text{with } f(0) = 1 \text{ and } f'(0) = 0.$$

4. Use the properties of sin and cos to find formulas for
   (a) $\sin(x - y)$;
   (b) $\cos(x - y)$.
5. Prove that

$$\cos(x + y) = \cos x \cos y - \sin y \sin x \quad \text{for all reals } x, y.$$

6. (a) Show that
$$\cos^2 x = \frac{\cos(2x)+1}{2}.$$
(b) Show that
$$\sin^2 x = \frac{1-\cos(2x)}{2}.$$
7. Find all the solutions of $\sin x = 0$.
8. Show that the series $\sum_{n=1}^{\infty}(1-\cos(1/n))$ converges.
9. Let a_n be a strictly positive sequence.
(a) Show that if the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} \sin a_n$ converges.
(b) Is the converse of (a) true?
10. In this exercise we will find the exact values of sin and cos for $\pi/3$.
(a) Show that
$$\cos 3x = 4\cos^3 x - 3\cos x.$$
(b) Let $a = \cos\pi/3$. Use (a) to show that
$$4a^3 - 3a + 1 = 0.$$
(c) Show that $a = \cos\pi/3 = 1/2$.
(d) Show that $\sin\pi/3 = \sqrt{3}/2$.
11. Use that $\pi/6 = \pi/2 - \pi/3$ and Exercise 10 to show that $\sin\pi/6 = 1/2$ and $\cos\pi/6 = \sqrt{3}/2$.
12. (a) Find all the positive solutions of the equation $\sin(1/x) = 0$.
(b) Is there a smallest positive solution of the equation $\sin(1/x) = 0$?
13. Let f be a continuous function such that $f(0) = 1$ and such that the equation $f(x) = 0$ has a smallest positive solution a. Show that $f(x) > 0$ for x in $[0, a]$.
14. Show that $\cos(x + 2\pi) = \cos(x)$.
15. Prove by induction that
$$1 - \frac{x^2}{2} + \cdots - \frac{x^{4k-2}}{(4k-2)!} \le \cos x \le 1 - \frac{x^2}{2} + \cdots - \frac{x^{4k-2}}{(4k-2)!} + \frac{x^{4k}}{(4k)!}$$
and
$$x - \frac{x^3}{3!} + \cdots - \frac{x^{4k-1}}{(4k-1)!} \le \sin x \le x - \frac{x^3}{3!} + \cdots - \frac{x^{4k-1}}{(4k-1)!} + \frac{x^{4k+1}}{(4k+1)!}$$
for any natural k and any real $x \ge 0$.
16. (a) Estimate $\cos 3$. Give a bound on the error.
(b) How many terms are needed in the power series expansion in order to have an error less than 0.001 for $\cos 3$.
17. If we use $x - x^3/3!$ to approximate $\sin x$ on [0, 1], find a bound on the error we make.

3.3 Inverse Trigonometric Functions

Recall that a function f from a set A to a set B is a relation between A and B such that for each element of A, f assigns exactly one element in B. The function is said to have an inverse f^{-1} if one can reverse the assignment. That is, f has an inverse if for every y in B, there is a unique solution x in A of the equation

$$f(x) = y.$$

If that is the case, we can define the inverse function f^{-1} by setting $f^{-1}(y) = x$ where x is the unique solution of $f(x) = y$. Note that for every y in B, we have

$$f(x) = f\big(f^{-1}(y)\big) = y$$

and that for every x in A, we have

$$f^{-1}(y) = f^{-1}\big(f(x)\big) = x.$$

For a function f to have an inverse, it must be one-to-one, that is, if $f(x) = f(y)$, then $x = y$. Since the trigonometric functions sin and cos are periodic, they cannot be one-to-one. For instance, $\sin(2\pi) = \sin(0)$, but $2\pi \neq 0$. However, if we restrict the domain, the function may become one-to-one. We start with sin.

Inverse of the Sine function
Consider sin on $[-\pi/2, \pi/2]$. Then, it has an inverse function denoted by arcsin which is defined on $[-1, 1]$ and differentiable on $(-1, 1)$. Its derivative is

$$(\arcsin)'(x) = \frac{1}{\sqrt{1 - x^2}} \quad \text{for all } x \in (-1, 1).$$

We first need the sign of $(\sin)' = \cos$ on $(-\pi/2, \pi/2)$. We have

$$\cos(x) > 0 \quad \text{for } x \in [0, \pi/2),$$

since $\cos(0) = 1$ and $\pi/2$ is defined as the smallest positive 0 of cos. Moreover, cos is even, so $\cos(x) = \cos(-x) > 0$ for x in $(-\pi/2, 0]$. Therefore, sin is strictly increasing on $(-\pi/2, \pi/2)$, and so is one-to-one. Let y be in $[-1, 1]$. Since

$$\sin(-\pi/2) = -1 \quad \text{and} \quad \sin(\pi/2) = 1$$

and sin is continuous on $[-\pi/2, \pi/2]$, we may apply the intermediate value theorem: there is x in $[-\pi/2, \pi/2]$ such that

$$\sin(x) = y.$$

Moreover, x is unique since sin is one-to-one on $[-\pi/2, \pi/2]$. Hence, we may define the function arcsin by

$$\arcsin(y) = x$$

for every x in $[-1, 1]$. In other words, sin has an inverse function that we denote by arcsin. It is defined on $[-1, 1]$. Observe that

$$\sin(x) = \sin\big(\arcsin(y)\big) = y \quad \text{for all } y \in [-1, 1]$$

and that

$$\arcsin(y) = \arcsin\big(\sin(x)\big) = x \quad \text{for all } x \in [-\pi/2, \pi/2].$$

That is, arcsin is the inverse function of sin. Recall from Calculus that if a function is differentiable on some open interval I and if f' is never 0 on I, then f^{-1} is defined on some interval J (which is the image interval of I) and is differentiable on J. Moreover, we have

$$\big(f^{-1}\big)' = \frac{1}{f' \circ f^{-1}}.$$

All this will be proved in Chap. 5.

In this particular case, $f = \sin$, $f' = \cos$, and $f^{-1} = \arcsin$, $I = (-\pi/2, \pi/2)$ and $J = (-1, 1)$. Thus,

$$(\arcsin)' = \frac{1}{\cos \circ \arcsin}.$$

For every x in $(-1, 1)$,

$$\cos^2\big(\arcsin(x)\big) + \sin^2\big(\arcsin(x)\big) = 1,$$

and since cos is strictly positive on $(-\pi/2, \pi/2)$, we have

$$\cos\big(\arcsin(x)\big) = \sqrt{1 - x^2}.$$

Thus,

$$(\arcsin)'(x) = \frac{1}{\sqrt{1 - x^2}} \quad \text{for all } x \in (-1, 1).$$

The preceding approach works very well for cos except that cos is not one-to-one on $[-\pi/2, \pi/2]$. Instead, we take $[0, \pi]$. The inverse function arccos is then defined on $[-1, 1]$ and is differentiable on $(-1, 1)$. The details are left as an exercise.

We define

Tangent function

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

This function is defined everywhere except on the set $\{\pi/2 + k\pi; k \in \mathbf{Z}\}$, which is the set where cos is 0. It has a period of π.

The domain and periodicity of tan will be found in the exercises. We now turn to the inverse function.

Inverse of the Tangent function
Consider tan on $(-\pi/2, \pi/2)$. Then, it has an inverse function denoted by arctan which is defined and differentiable on $\mathbf{R}$. Its derivative is

$$(\arctan)'(x) = \frac{1}{1+x^2} \quad \text{for all } x \in \mathbf{R}.$$

Consider tan on $(-\pi/2, \pi/2)$. Then, using the quotient rule for differentiation,

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cos x - (-\sin x)(\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} > 0.$$

That is, tan is strictly increasing on $(-\pi/2, \pi/2)$. The range of tan is all $\mathbf{R}$. We will not prove this but only give the ideas of the proof. A formal proof is left as an exercise. As x approaches $-\pi/2$ from the right, $\cos x$ approaches 0 from the positive side, and sin approaches -1. Therefore, $\tan x$ goes to $-\infty$. Similarly, as x approaches $\pi/2$ from the left, $\tan x$ approaches $+\infty$. Since tan is continuous, all the intermediate values are attained (by the intermediate value theorem). Thus, the inverse function of tan, arctan, is defined on $\mathbf{R}$. The function arctan is differentiable on $\mathbf{R}$, and we have

$$(\arctan)' = \frac{1}{(\tan)' \circ \arctan} = \frac{1}{\frac{1}{\cos^2} \circ \arctan} = \cos^2 \circ \arctan.$$

Note that for any x in $\mathbf{R}$,

$$\cos^2(\arctan x) + \sin^2(\arctan x) = 1$$

and, dividing by $\cos^2(\arctan x)$,

$$1 + \tan^2(\arctan x) = \frac{1}{\cos^2(\arctan x)},$$

and so

$$\frac{1}{\cos^2(\arctan x)} = 1 + x^2.$$

Therefore, for all x in $\mathbf{R}$, we have

$$(\arctan)'(x) = \frac{1}{1+x^2}.$$

Exercises

1. Prove that the function tan has period π.
2. Find the domain of tan.
3. Sketch the graph of tan.
4. Assume that f is continuous on the interval I.
   (a) Show that if f is strictly increasing on I, then it is one-to-one.
   (b) Is the converse of (a) true?

5. Find a numerical approximation for $\tan 1$.
6. (a) Show that cos, restricted to $[0, \pi]$, has an inverse function.
   (b) Show that arccos is differentiable on $(-1, 1)$ and that
$$(\arccos)'(x) = -\frac{1}{\sqrt{1-x^2}}.$$
   (c) Show that for any x in $[-1, 1]$, we have
$$\arccos(x) + \arcsin(x) = \pi/2.$$
7. In this exercise we show that the range of tan is **R**.
   (a) Consider the sequence $\pi/2 - 1/n$. Show that $\cos a_n$ converges to 0 and $\sin a_n$ converges to 1.
   (b) Let $A > 0$. Show that there is N such that for $n \geq N$, we have
$$\cos a_n < 1/(2A) \quad \text{and} \quad \sin a_n > 1/2.$$
   (c) Show that for $n \geq N$, $\tan a_n > A$.
   (d) Show that for any a in $[0, A]$, the equation $\tan x = a$ has a solution.
   (e) Show that for any $a \geq 0$, the equation $\tan x = a$ has a solution.
   (f) Show that for any real a, the equation $\tan x = a$ has a solution. This proves that the range of tan is **R**.

3.4 Exponential and Logarithmic Functions

We use a power series to define the exponential function.

Exponential function
The exponential function exp is defined for all real x by
$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
In particular, $\exp(0) = 1$.

We need to check that the power series above converges for every x in **R**. Let
$$a_n = \left|\frac{x^n}{n!}\right|.$$
Then
$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!}|x| = \frac{|x|}{n+1},$$
which converges to $0 < 1$ for any fixed x as n goes to infinity. By the ratio test, the series converges absolutely for any fixed x, and the function exp is defined on all of **R**.

Letting $x = 0$ in the series yields $\exp(0) = 1$. We now state an important property.

Multiplicative property
For any reals x and y, we have

$$\exp(x + y) = \exp(x)\exp(y).$$

In particular,

$$\exp(-x) = \frac{1}{\exp(x)} \quad \text{for all } x.$$

We first need to argue that exp is a differentiable function. *If* we were allowed to differentiate the infinite series term by term, we would get

$$(\exp)'(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right)' = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots.$$

That is, it appears that the derivative of exp is itself. As pointed out before, this is a consequence of a nontrivial result (power series may be differentiated term by term on an open interval of convergence). We will prove this in Chap. 7. Hence, exp is differentiable on $\mathbf{R}$, and

$$(\exp)' = \exp.$$

Back to the multiplicative property. We start by proving that

$$\exp(-x) = \frac{1}{\exp(x)} \quad \text{for all } x.$$

Let h be defined by

$$h(x) = \exp(x)\exp(-x).$$

Since exp is differentiable, so is h (a product of differentiable functions is differentiable). By the product and chain rules, we have

$$h'(x) = \exp(x)\exp(-x) - \exp(x)\exp(-x) = 0.$$

Since h' is identically 0 on $\mathbf{R}$, h is a constant. Since $\exp(0) = 1$, we have that h is the constant $h(0) = 1$. That is,

$$\exp(x)\exp(-x) = 1 \quad \text{for all } x \in \mathbf{R}.$$

Note that this is a particular case of the multiplicative property above.

In order to prove the general case of the multiplicative property, we will use the lemma below.

Lemma 3.1 *Let f be a function which is differentiable on* $\mathbf{R}$ *and such that*

$$f' = f$$

and $f(0) = C$. *Then,*

$$f(x) = C \exp(x) \quad \textit{for all } x \in \mathbf{R}.$$

We now prove Lemma 3.1. Let f be a solution of the differential equation $f' = f$, and let g be defined by

$$g(x) = f(x)\exp(-x).$$

Since f and exp are differentiable, so is g. By the product and chain rules we get

$$g'(x) = f'(x)\exp(-x) - f(x)\exp(-x) = f(x)\exp(-x) - f(x)\exp(-x) = 0.$$

That is, g' is identically 0. Therefore, g is a constant. Since $f(0) = C$ and $\exp(0) = 1$, we have $g = g(0) = C$. Hence, $f(x) = C\exp(x)$ for every x in $\mathbf{R}$. Lemma 3.1 is proved.

We turn to the proof of the multiplicative property. Fix a real a and define the function k by $k(x) = \exp(a + x) - \exp(a)\exp(x)$. The function k is differentiable, and

$$k'(x) = \exp(a + x) - \exp(a)\exp(x) = k(x).$$

That is, $k' = k$. By Lemma 3.1, $k(x) = k(0)\exp(x)$ for every x. But $k(0) = 0$. So

$$\exp(a + x) - \exp(a)\exp(x) = 0$$

for all a and all x. This proves the multiplicative property.

The next properties involve limits at infinity. We start by giving two formal definitions.

Finite limit at infinity
A function f defined on the reals is said to have a limit ℓ at positive infinity if for any $\epsilon > 0$, there is B such that if $x > B$, then $|f(x) - \ell| < \epsilon$. The notation is

$$\lim_{x \to +\infty} f(x) = \ell.$$

If the limit is ℓ at $-\infty$, then for any $\epsilon > 0$, there must be a B such that if $x < B$, then $|f(x) - \ell| < \epsilon$.

Infinite limit at infinity
A function f defined on the reals is said to go to positive infinity at positive infinity if for any $A > 0$, there is B such that if $x > B$, then $f(x) > A$. The notation is

$$\lim_{x \to +\infty} f(x) = +\infty.$$

Note that a function that tends to positive (or negative) infinity cannot be bounded. For any A, it is possible to find an x such that $f(x) > A$. However, not all unbounded functions tend to infinity: the function may have arbitrarily large oscillations. In the exercises such an example is provided.

Properties of the exponential function

(i) $\exp(x) > 0$ for all $x \in \mathbf{R}$.

For every integer n, we have

(ii) $$\lim_{x\to+\infty} \frac{\exp(x)}{x^n} = +\infty.$$

(iii) $$\lim_{x\to+\infty} x^n \exp(-x) = 0.$$

(iv) The function exp is strictly increasing on $\mathbf{R}$.

Note that (ii) implies that $\exp(x)$ grows much faster than any power function x^n as x goes to infinity. Observe also, by (iii), that $\exp(-x)$ decreases to 0 much faster than any power function x^{-n} as x goes to $+\infty$.

We start by proving (i). Take $x \geq 0$. Then

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \geq 1.$$

Thus, $\exp(x) > 0$ for every $x \geq 0$. Take $x < 0$. Then $\exp(-x) > 0$, and

$$\exp(x) = \frac{1}{\exp(-x)} > 0,$$

and so $\exp(x) > 0$ as well. That is, for any x in $\mathbf{R}$, we have that $\exp(x) > 0$, and (i) is proved.

We now turn to (ii). Fix a natural $n \geq 1$. Let $A > 0$ be given. Set $B = (n!A)^{1/n}$. Observe that

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots > \frac{x^n}{n!} \quad \text{for } x > 0.$$

Thus, if $x > B$, we have

$$\frac{x^n}{n!} > \frac{B^n}{n!} = A.$$

Hence,

$$\exp(x) > \frac{x^n}{n!} > A.$$

This proves (ii) for $n \geq 1$. If $n \leq 0$, then

$$x^{-n} \geq 1 \quad \text{for } x \geq 1$$

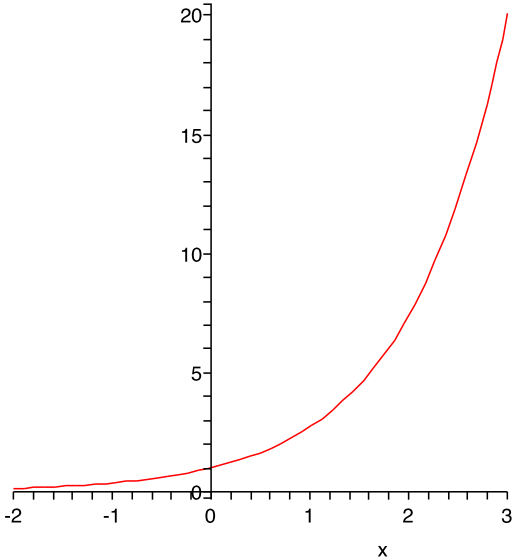

Fig. 3.3 This is the graph of the exponential function

and

$$\frac{\exp(x)}{x^n} \geq \exp(x) \quad \text{for } x \geq 1.$$

Hence, (ii) must hold for $n \leq 0$ as well (see the exercises).

The statement (iii) is a direct application of the following lemma.

Lemma 3.2 *If* $\lim_{x\to+\infty} f(x) = +\infty$, *then* $\lim_{x\to+\infty} \frac{1}{f(x)} = 0$.

We now prove Lemma 3.1. Let $\epsilon > 0$. Since f goes to infinity at infinity, we have a B such that if $x > B$, then

$$f(x) > 1/\epsilon.$$

Hence,

$$\frac{1}{f(x)} = \left|\frac{1}{f(x)} - 0\right| < \epsilon.$$

This proves Lemma 3.2. The proof of (iii) is left to the reader.

Property (iv) comes from the fact that $(\exp)' = \exp$ and that exp is strictly positive on the reals. Hence, it is strictly increasing on the reals.

Figure 3.3 is the graph of the function exp. We now indicate how to sketch this graph. The derivative of exp is itself, and exp is always positive. This implies that exp is strictly increasing on $\mathbf{R}$. Since the second derivative of exp is also exp, it is also concave up. Moreover, we know that $\exp(0) = 1$. Finally, the limit of $\exp(x)$ as x to $-\infty$ is the same as the limit of $\exp(-x)$ as x goes to $+\infty$ (why?). This limit has been shown to be 0 in (iii).

Logarithmic function
The function exp has an inverse function denoted by ln. The logarithmic function ln is defined on $(0, \infty)$, and we have

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad \text{for all } x > 0.$$

In particular, $\ln 1 = 0$, and ln is increasing on $(0, \infty)$.

We first need to explain why the exponential function has an inverse function. For any given $y > 0$, the equation

$$\exp(x) = y$$

has a unique solution. The existence of the solution is a consequence of the intermediate value theorem: since exp is continuous, $\exp(x)$ tends to 0 as x goes to $-\infty$ and tends to $+\infty$ as x goes to $+\infty$, all the reals strictly larger than 0 must be attained. The details will be given as an exercise. The uniqueness of the solution is a consequence of the fact that exp is one-to-one: since $(\exp)' = \exp$, the function exp is strictly increasing on $\mathbf{R}$. Therefore, it is one-to-one. By setting $\ln y = x$ where x is the unique solution of $\exp(x) = y$, we define the inverse function of exp, denoted by ln, on $(0, \infty)$.

We now prove that the inverse function ln can actually be defined by $\ln x = \int_1^x \frac{1}{t}\,dt$ for $x > 0$. Since exp is differentiable on the open interval $\mathbf{R}$ and its derivative is never 0, its inverse function ln is differentiable on $(0, \infty)$. Using that $(\exp)' = \exp$, we have

$$(\ln)'(x) = \frac{1}{\exp(\ln(x))} = \frac{1}{x}.$$

Let

$$F(x) = \int_1^x \frac{1}{t}\,dt \quad \text{for all } x > 0.$$

Then, by the fundamental theorem of Calculus, since $1/x$ is continuous on $(0, \infty)$, F is differentiable on the same interval, and

$$F'(x) = 1/x.$$

Therefore,

$$F' = (\ln)' \quad \text{on } (0, \infty).$$

Thus, $F - \ln$ is a constant. Using that $\exp(0) = 1$, we have $\ln 1 = 0$. Since $F(1) = \ln 1 = 0$, this constant must be 0. Therefore,

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad \text{for all } x > 0.$$

Using again that

$$(\ln)'(x) = \frac{1}{x}$$

for all $x > 0$ and that $1/x > 0$ for $x > 0$, we get that ln is increasing on $(0, \infty)$.

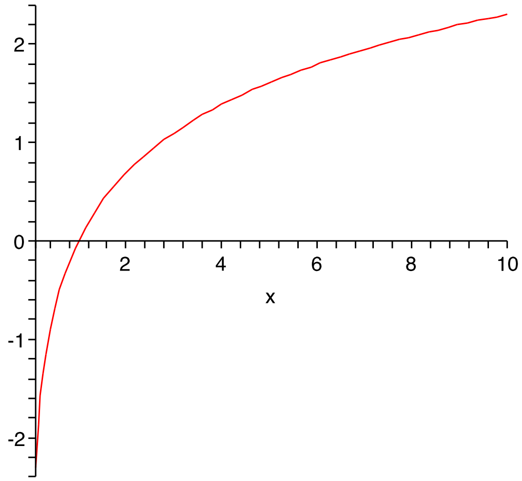

Fig. 3.4 This is the graph of the logarithmic function

Limits for the logarithmic function
We have

$$\text{(v)} \qquad \lim_{x\to+\infty} \ln x = +\infty,$$

$$\text{(vi)} \qquad \lim_{x\to 0^+} \ln x = -\infty,$$

and for every rational $r > 0$,

$$\text{(vii)} \qquad \lim_{x\to+\infty} \frac{\ln x}{x^r} = 0.$$

Note that (vii) states that ln goes to infinity much slower than any positive power function. It will be proved in the exercises.

We first prove (v). Let $A > 0$ be a real, and let $B = \exp(A)$. If $x > B$, then $\ln x > \ln B = A$. Hence, the limit at infinity of ln is infinite, and that proves (v).

We now turn to the limit at 0^+. The function ln is only defined on the positive reals, and so 0 can be approached only from the right. In order to prove that the limit of ln at 0^+ is $-\infty$, we need to show that for any $A > 0$, there is a $\delta > 0$ such that if $0 < x < \delta$, then $\ln x < -A$. For a fixed A, let $\delta = \exp(-A)$. If $0 < x < \delta$, then

$$\ln x < \ln \delta = -A.$$

Hence, the limit of ln at 0^+ is $-\infty$. This proves (vi).

Figure 3.4 is the graph of the ln function. Its derivative $1/x$ is positive on the positive reals. Hence, it is an increasing function. Its second derivative is $-1/x^2$, and so ln is concave down. The limits in (v) and (vi) are also useful.

The following lemma will be useful to prove more properties of the ln function.

Lemma 3.3 *Let a be a real, and r a rational. Then*

$$\big(\exp(a)\big)^r = \exp(ar).$$

We first prove Lemma 3.3 for r natural. For $r = 1$, the equality obviously holds. Assume that it holds for r. We have

$$\big(\exp(a)\big)^{r+1} = \big(\exp(a)\big)^r \exp(a) = \exp(ar)\exp(a) = \exp(ar + a) = \exp\big(a(r + 1)\big),$$

where the first equality comes from the definition of natural powers, the second is the induction hypothesis, and the third is the multiplicative property of the exp function. Hence, the formula holds for $r + 1$. Therefore, the formula in Lemma 3.3 is proved, by induction, for r natural.

Let n be a natural, and let $r = 1/n$. Then

$$\big(\exp(ar)\big)^n = \exp(arn),$$

where we are using the formula for a natural n and where ar plays the role of a. Since $rn = 1$, we have

$$\big(\exp(ar)\big)^n = \exp(a).$$

Hence,

$$\Big(\big(\exp(ar)\big)^n\Big)^{1/n} = \big(\exp(a)\big)^{1/n}.$$

That is,

$$\exp(ar) = \big(\exp(a)\big)^{1/n}.$$

Therefore, Lemma 3.3 holds for $r = 1/n$. Assume now that $r = p/q$ where p and q are naturals. We have

$$\begin{aligned}\big(\exp(a)\big)^r &= \big(\exp(a)\big)^{p/q} = \Big(\big(\exp(a)\big)^p\Big)^{1/q} = \big(\exp(pa)\big)^{1/q}\\ &= \exp(pa/q) = \exp(ra),\end{aligned}$$

where the third equality comes from the formula for naturals, and the fourth from the formula for inverse of naturals. This proves Lemma 3.3 for positive rationals. The proof for negative rationals is an easy consequence left to the reader.

For a real $a > 0$, we have

$$a = \exp(\ln a).$$

Hence, by Lemma 3.3,

$$a^r = \big(\exp(\ln a)\big)^r = \exp(r \ln a),$$

where r is a rational. We use this observation to define a^x for real x.

Defining real powers

For any $a > 0$ and any x in $\mathbf{R}$, we define

$$a^x = \exp(x \ln a).$$

Note that before this definition we could not make sense of $3^{\sqrt{2}}$. Now we have defined this as $\exp(\sqrt{2}\ln 3)$.

We now turn to additional properties of ln.

Properties of the logarithmic function

(a) $\ln(xy) = \ln(x) + \ln(y)$ for all $x > 0$ and $y > 0$.
(b) $\ln(a^x) = x\ln(a)$ for all $x \in \mathbf{R}$ and $a > 0$.

Property (a) is a direct consequence of the multiplicative property of exp. Given $x > 0$ and $y > 0$, let

$$s = \ln x \quad \text{and} \quad \ln y = t.$$

Then,

$$\exp(s+t) = \exp(s)\exp(t) = xy.$$

Taking ln on both sides yields

$$\ln\big(\exp(s+t)\big) = \ln(xy).$$

Thus,

$$s + t = \ln(x) + \ln(y) = \ln(xy).$$

Property (a) is proved.

We now turn to (b). By the definition of a^x, we have

$$a^x = \exp(x\ln a).$$

Hence,

$$\ln(a^x) = \ln\big(\exp(x\ln a)\big) = x\ln a.$$

This proves (b).

The number e

We define the number e as $e = \exp(1)$. That is,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Application 3.2 Evaluate the number e.

We start by evaluating e^{-1}. We have

$$e^{-1} = \frac{1}{\exp(1)} = \exp(-1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}.$$

The preceding series satisfies the conditions of the alternating series theorem (the sequence $a_n = 1/n!$ is positive, decreasing, and converges to 0). Taking the first 10 and 11 terms, we get

$$\sum_{n=0}^{9} \frac{(-1)^n}{n!} = \frac{16687}{45360}$$

and

$$\sum_{n=0}^{10} \frac{(-1)^n}{n!} = \frac{16481}{44800}.$$

Since this is an alternating series, we have the following bounds:

$$\frac{16687}{45360} < e^{-1} < \frac{16481}{44800}.$$

Thus,

$$\frac{44800}{16481} < e < \frac{45360}{16687}.$$

In particular, we get the first five decimals of e: 2.71828.

The importance of the number e comes from the following property.

A new expression for the exponential function
We have, for any x real, that

$$\exp(x) = e^x.$$

Recall that for any $a > 0$ we defined

$$a^x = \exp(x \ln a).$$

We use this for $a = e$ to get

$$e^x = \exp(x \ln e).$$

Since $\exp(1) = e$, we have that $\ln e = 1$. Thus,

$$e^x = \exp(x \ln e) = \exp(x),$$

which proves the formula.

Application 3.3 The number e is irrational.

We do a proof by contradiction. Assume that e is a rational p/q. Since e is strictly between 2 and 3, it is not a natural, and $q > 1$. We have

$$q!e = q!\frac{p}{q} = (q-1)!p,$$

and so $q!e$ is a natural number. By definition

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \sum_{n=0}^{q} \frac{1}{n!} + \sum_{n=q+1}^{\infty} \frac{1}{n!}.$$

Observe that

$$q! \sum_{n=0}^{q} \frac{1}{n!} = q! + q! + q!/2 + q!/3! + \cdots + q!/q!.$$

Every $q!/n!$ for $n = 0, 1, \dots, q$ is a natural (why?). Hence, $q! \sum_{n=0}^{q} \frac{1}{n!}$ is natural as well. On the other hand,

$$q! \sum_{n=q+1}^{\infty} \frac{1}{n!} = q!/(q+1)! + q!/(q+2)! + q!/(q+3)! + \cdots.$$

We have

$$q!/(q+1)! = 1/(q+1) < 1/2 \quad \text{and} \quad q!/(q+2)! = 1/(q+1)(q+2) < 1/2^2.$$

More generally,

$$q!/(q+k)! < 1/2^k \quad \text{for } k = 1, 2, \dots.$$

Hence,

$$q! \sum_{n=q+1}^{\infty} \frac{1}{n!} < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

In summary, $q!e$ is a natural, $q! \sum_{n=0}^{q} \frac{1}{n!}$ is also a natural, but

$$q! \sum_{n=q+1}^{\infty} \frac{1}{n!} = q!e - q! \sum_{n=0}^{q} \frac{1}{n!}$$

is strictly between 0 and 1. The difference of two natural numbers must be an integer and cannot be in $(0, 1)$. We have a contradiction. The number e cannot be a rational.

Exercises

1. (a) Show that for any $x > 0$ and any natural n, we have

$$\exp\left(\sqrt{x}\right) > \frac{x^{n+1}}{(2n+2)!}.$$

   (b) Prove that

$$\lim_{x \to +\infty} \frac{\exp(\sqrt{x})}{x^n} = +\infty.$$

   (c) Generalize (b).
2. Use Lemma 3.1 to prove (iii).

3. (a) Let $s > 0$ be a rational. Show that if $t > 1$, then
$$t^{-1} < t^{-1+s}.$$
(b) Integrate both sides of (a) for t between 1 and $x > 1$ to get
$$\ln x \leq \frac{x^s - 1}{s}.$$
(c) Compute the limit in (vii).
4. Let $a > 0$ and define $f(x) = a^x$ for every x. Compute the derivative of f.
5. Prove Lemma 3.2 for negative rationals.
6. Let f be a continuous function on $\mathbf{R}$. Assume that
$$\lim_{x \to +\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = 0.$$
We are going to show that for a given $y > 0$, the equation $f(x) = y$ has at least one solution.
(a) Show that there is $C < 0$ such that if $x < C$, then $f(x) < y/2$.
(b) Show that there is $A > 0$ such that if $x > A$, then $f(x) > 2y$.
(c) Show that the equation $f(x) = y$ has at least one solution.
7. Find the first 12 decimals of the number e.
8. Show that the function $f(x) = x\cos(\pi x/2)$ is unbounded but does not tend to infinity as x tends to positive infinity.
9. Take a real x.
(a) Show that
$$\ln\big((1 + x/n)^n\big) = x\frac{\ln(1 + x/n) - \ln 1}{x/n}.$$
(b) Show that
$$\lim_{n \to \infty} \ln\big((1 + x/n)^n\big) = x.$$
(c) Prove that
$$\lim_{n \to \infty} (1 + x/n)^n = e^x.$$
(d) Use (c) to find an estimate for e.
10. In this exercise we are going to show that the multiplicative property characterizes exponential functions. That is, the only continuous functions having the multiplicative property are the functions $\exp(ax)$ for a real a and the zero function. Assume that f is continuous with $f(1) \neq 0$ and is such that for all reals x and y,
$$f(x + y) = f(x)f(y).$$
(a) Show that $f(1) > 0$.
(b) For every natural n, $f(n) = f(1)^n$.
(c) For every natural n, $f(1/n) = f(1)^{1/n}$.
(d) For every rational r, $f(r) = f(1)^r$.

(e) Show that there is a real a such that for all real x,

$$f(x) = \exp(ax).$$

11. Assume that f is continuous, with $f(1) = 0$ and is such that for all reals x and y

$$f(x+y) = f(x)f(y).$$

Show that f is identically 0. (Follow the pattern of the preceding exercise.)

12. Does the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

converge?

Chapter 4
Fifty Ways to Estimate the Number π

4.1 Power Series Expansions

The number π has been defined in a rather abstract form so far: we have shown that sin and cos are periodic functions and that their period is defined to be 2π. In this chapter we will use several methods to estimate the number π. Estimating π is important for applications (it appears in all kinds of mathematical formulas in geometry, physics, probability, and so on) and also from a theoretical point of view. What is this number? More generally, what is a number? What is its exact value? Why do we need approximations?

The first methods we will see rely on power series expansions.

Power series expansion for $\frac{1}{1-x}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for all } x \in (-1, 1).$$

We start with the algebraic formula

$$(1-x)\big(1 + x + x^2 + \cdots + x^n\big) = 1 - x^{n+1}.$$

Thus, for $x \neq 1$,

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}. \tag{4.1}$$

If $|x| < 1$, then we know that

$$\lim_{n\to\infty} x^{n+1} = 0.$$

For fixed x in $(-1, 1)$, we let n go infinity in (4.1) to get

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1.$$

R.B. Schinazi, *From Calculus to Analysis*,
DOI 10.1007/978-0-8176-8289-7_4,

This proves the formula above.

The algebraic formula (4.1) can be used to derive a number of other useful power series expansions as we will now see. We will need the following facts from Calculus regarding integrals and inequalities.

Inequalities and integrals

Assume that f and g are continuous on $[a, b]$, $a < b$.

(S1) If $f(x) \leq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx,$$

(S2) If $f(x) \geq 0$ for all x in $[a, c]$ and $b < c$, then

$$\int_a^b f(x)\,dx \leq \int_a^c f(x)\,dx,$$

(S3)
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b \left|f(x)\right|dx,$$

(S4)
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Note in particular that if f is continuous and positive on $[a, b]$, then by $S1$ we have that

$$\int_a^b f(x)\,dx \geq \int_a^b 0\,dx = 0.$$

We now turn to our second power series expansion. We make the substitution $x = -y^2$ into (4.1) to get

$$1 - y^2 + y^4 - \cdots + (-1)^n y^{2n} = \frac{1 - (-y^2)^{n+1}}{1 + y^2}.$$

On both sides of the equality we have continuous functions that we now integrate between 0 and $x > 0$. This yields

$$\int_0^x \left(1 - y^2 + y^4 - \cdots + (-1)^n y^{2n}\right) dy = \int_0^x \left(\frac{1 - (-y^2)^{n+1}}{1 + y^2}\right) dy.$$

Thus,

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} = \int_0^x \frac{1}{1 + y^2}\,dy + (-1)^{n+2} \int_0^x \frac{y^{2n+2}}{1 + y^2}\,dy.$$

Recall that

$$\int_0^x \frac{1}{1 + y^2}\,dy = \arctan x.$$

Hence,

$$\left|x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} - \arctan x\right| = \left|(-1)^{n+2} \int_0^x \frac{y^{2n+2}}{1+y^2}\, dy\right|.$$

We need to show that for fixed x in $[-1, 1]$, the right-hand side goes to 0 as n goes to infinity.

First, fix x in $[0, 1]$. Since $y^{2n+2} \geq 0$, we have

$$\left|(-1)^{n+2} \int_0^x \frac{y^{2n+2}}{1+y^2}\, dy\right| = \int_0^x \frac{y^{2n+2}}{1+y^2}\, dy.$$

Using that $1 + y^2 \geq 1$ for all y,

$$\frac{1}{1+y^2} \leq 1,$$

and so

$$\frac{y^{2n+2}}{1+y^2} \leq y^{2n+2}.$$

Since $0 \leq x \leq 1$ and the functions on both sides of the inequality are continuous,

$$\int_0^x \frac{y^{2n+2}}{1+y^2}\, dy \leq \int_0^x y^{2n+2}\, dy \leq \int_0^1 y^{2n+2}\, dy = \frac{1}{2n+3},$$

where we use (S1) for the first inequality and (S2) for the second.

We now turn to x in $[-1, 0]$:

$$\left|(-1)^{n+2} \int_0^x \frac{y^{2n+2}}{1+y^2}\, dy\right| = \left|\int_0^x \frac{y^{2n+2}}{1+y^2}\, dy\right| = \int_x^0 \frac{y^{2n+2}}{1+y^2}\, dy,$$

where the last equality comes from (S4). By (S2) we get

$$\int_x^0 \frac{y^{2n+2}}{1+y^2}\, dy \leq \int_{-1}^0 \frac{y^{2n+2}}{1+y^2}\, dy \leq \int_{-1}^0 y^{2n+2} = \frac{1}{2n+3}.$$

Therefore, we have for any x in $[-1, 1]$,

$$\left|x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} - \arctan x\right| \leq \frac{1}{2n+3}.$$

For fixed x in $[-1, 1]$, we let n go to infinity, and we get the following power series expansion.

Power series expansion for arctan

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for all } x \in [-1, 1].$$

Note that, unlike what happens for $\frac{1}{1-x}$, the power series expansion for arctan converges at the end points.

Application 4.1 We use the formula above to estimate π. Letting $x = 1$ and using that $\arctan(1) = \pi/4$, we get

$$\pi/4 = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}.$$

We sum up to $n = 1000$ to get the approximation 3.142591654 for π. Only the first two decimals are right! This is due to the fact that the series converges rather slowly. As seen above, the error in summing up to n is less than $1/(2n+3)$. In the present case the error is less than $1/2003$.

The same power series expansion converges much faster for an $x < 1$ as we will see next.

Application 4.2 Recall that

$$\tan(\pi/6) = \frac{1}{\sqrt{3}}.$$

Thus, $\arctan(\frac{1}{\sqrt{3}}) = \pi/6$. The power series expansion of arctan at $x = \frac{1}{\sqrt{3}}$ is going to converge much faster than at 1 because $\frac{1}{\sqrt{3}} < 1$. We have

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{\sqrt{3}})^{2n+1}}{2n+1}.$$

Since

$$\left(\frac{1}{\sqrt{3}}\right)^{2n+1} = \frac{1}{\sqrt{3}}\frac{1}{3^n},$$

we have

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}(-1)^n \frac{1}{3^n(2n+1)}.$$

Note that the alternating series theorem applies to $\sum_{n=0}^{\infty}(-1)^n \frac{1}{3^n(2n+1)}$ (why?). Thus,

$$\sum_{n=0}^{9}(-1)^n \frac{1}{3^n(2n+1)} \le \sum_{n=0}^{\infty}(-1)^n \frac{1}{3^n(2n+1)} \le \sum_{n=0}^{10}(-1)^n \frac{1}{3^n(2n+1)}.$$

This yields the numerical bounds

$$0.90689906 \le \sum_{n=0}^{\infty}(-1)^n \frac{1}{3^n(2n+1)} \le 0.90689987.$$

We also have the following bounds for $\frac{1}{\sqrt{3}}$:

$$0.57735026 \le \frac{1}{\sqrt{3}} \le 0.57735027.$$

Thus,

$$0.5235984081 \le \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n(2n+1)} \le 0.5235988848.$$

The infinite series above is $\pi/6$, and multiplying by 6 both bounds yields the following estimate for π:

$$3.141590449 \le \pi \le 3.141592654.$$

That is, we get the first 5 decimals of π by summing only 11 terms!

We will now find a power series expansion for $\ln(1-x)$. Our starting point again is (4.1):

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x} \quad \text{for } x \neq 1.$$

We integrate between 0 and y to get

$$\int_0^y \left(1 + x + x^2 + \cdots + x^n\right) dx = \int_0^y \frac{1}{1-x}\, dx - \int_0^y \frac{x^{n+1}}{1-x}\, dx.$$

Therefore,

$$\sum_{k=0}^{n} \frac{y^{k+1}}{k+1} = -\ln(1-y) - \int_0^y \frac{x^{n+1}}{1-x}\, dx.$$

We now bound the remainder. First assume that y is in [0,1). Then

$$\frac{1}{1-x} \le \frac{1}{1-y} \quad \text{for } x \in [0, y].$$

Thus,

$$\left| -\int_0^y \frac{x^{n+1}}{1-x}\, dx \right| = \int_0^y \frac{x^{n+1}}{1-x}\, dx \le \frac{1}{1-y} \int_0^y x^{n+1}\, dx = \frac{1}{1-y} \frac{y^{n+2}}{n+2}.$$

Since for fixed y in [0, 1), $\frac{1}{1-y}\frac{y^{n+2}}{n+2}$ converges to 0 as n goes to infinity, we get the following power series expansion for y in [0, 1):

$$\ln(1-y) = -\sum_{k=0}^{\infty} \frac{y^{k+1}}{k+1}.$$

We now take care of negative y. Assume that y is in $[-1, 0]$. By (S4) and (S3) we have

$$\left| -\int_0^y \frac{x^{n+1}}{1-x}\, dx \right| = \left| \int_y^0 \frac{x^{n+1}}{1-x}\, dx \right| \le \int_y^0 \frac{|x|^{n+1}}{1-x}\, dx.$$

Since $x < 0$, we have $|x| = -x$ and

$$\int_y^0 \frac{|x|^{n+1}}{1-x}\, dx = \int_y^0 \frac{(-x)^{n+1}}{1-x}\, dx \le (-1)^{n+1} \int_y^0 x^{n+1}\, dx,$$

where we use that

$$\frac{1}{1-x} \le 1 \quad \text{for } x \le 0$$

and that

$$(-1)^{n+1} x^{n+1} \ge 0$$

for $x \le 0$ (why?). Thus,

$$\left| -\int_0^y \frac{x^{n+1}}{1-x}\, dx \right| \le (-1)^{n+2} \frac{y^{n+2}}{n+2}.$$

For fixed y in $[-1, 0)$,

$$\lim_{n\to\infty} (-1)^{n+2} \frac{y^{n+2}}{n+2} = 0.$$

We have completed the proof of the following power series expansion.

Power series expansion for $\ln(1-x)$

$$\ln(1-x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \quad \text{for all } x \in [-1, 1).$$

Application 4.3 Find a power series expansion for $\ln(\frac{1+x}{1-x})$.

Observe first that

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x).$$

Since

$$\ln(1-x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \quad \text{for all } x \in [-1, 1),$$

if we let $y = -x$, we get

$$\ln(1+y) = \sum_{k=0}^{\infty} (-1)^k \frac{y^{k+1}}{k+1} \quad \text{for all } y \in (-1, 1],$$

where we used that $(-1)^{k+2} = (-1)^k$. Subtracting the two power series, we get

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.$$

The even powers cancel, and we get

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \quad \text{for all } x \in (-1, 1).$$

Application 4.4 We know that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. In fact, since the partial sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

is an increasing sequence and it does not converge, it must go to infinity. In this application we will estimate how fast this happens. Define, for $k \geq 2$,

$$a_k = \ln k - \ln(k-1) - \frac{1}{k}.$$

Let

$$A_n = \sum_{k=2}^{n} a_k.$$

Note that

$$\sum_{k=2}^{n} \big(\ln k - \ln(k-1)\big) = \ln n - \ln 1 = \ln n.$$

Hence,

$$A_n = \ln n - \sum_{k=2}^{n} \frac{1}{k}.$$

We now show that A_n converges. Observe that

$$\ln n - \ln(n-1) = -\ln\left(\frac{n-1}{n}\right) = \ln\left(1 - \frac{1}{n}\right).$$

Using the power series expansion for $\ln(1-x)$ at $x = 1/n$, we get

$$\ln\left(1 - \frac{1}{n}\right) = -\sum_{k=1}^{\infty} \frac{1}{kn^k}.$$

Hence,

$$a_n = -\ln(1 - 1/n) - 1/n = \sum_{k=1} \frac{1}{kn^k} - \frac{1}{n} = \sum_{k=2} \frac{1}{kn^k} = \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots.$$

We have

$$\begin{aligned} a_n &= \sum_{k=2} \frac{1}{kn^k} < \sum_{k=2} \frac{1}{2n^k} = \frac{1}{2n^2} + \frac{1}{2n^3} + \cdots \\ &= \frac{1}{2n^2}\left(1 + \frac{1}{n} + \frac{1}{n^2} + \cdots\right) = \frac{1}{2n^2} \frac{1}{1 - 1/n}, \end{aligned}$$

where the last equality comes from the sum of a geometric series with ratio $r = 1/n$. On the other hand, it is clear that

$$a_n = \sum_{k=2} \frac{1}{kn^k} > \frac{1}{2n^2}.$$

Hence,

$$\frac{1}{2n^2} < a_n < \frac{1}{2n^2}\frac{1}{1-1/n}.$$

By the squeezing principle we get

$$\lim_{n\to\infty}\frac{a_n}{1/(2n^2)} = 1.$$

By the limit comparison test the series $\sum_{n\geq 2} a_n$ converges since $\sum_{n\geq 1} 1/(2n^2)$ converges. Therefore, the sequence

$$A_n = \ln n - \sum_{k=2}^{n}\frac{1}{k}$$

converges, and so

$$\ln n - \sum_{k=1}^{n}\frac{1}{k}$$

converges as well (why?). The limit of this last sequence is called Euler's constant. In summary, $\ln n$ and $\sum_{k=1}^{n}\frac{1}{k}$ go to infinity at the same speed (very slowly).

Up to this point all the power series expansions we have considered were derived from the geometric series. We now turn to a different method.

Binomial power series
Let α be a real number. Then, for all x in $(-1, 1)$,

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}x^k.$$

That is, for x in $(-1, 1)$, we have

$$(1+x)^\alpha = 1 + \alpha x + \alpha(\alpha-1)\frac{x^2}{2!} + \alpha(\alpha-1)(\alpha-2)\frac{x^3}{3!} + \cdots.$$

In order to establish the binomial expansion, we use a differential equation. Let

$$f(x) = (1+x)^\alpha \quad \text{for } x \in (-1, 1).$$

The function f is differentiable, and

$$f'(x) = \alpha(1+x)^{\alpha-1}.$$

Hence,

$$(1+x)f'(x) = \alpha(1+x)^\alpha = \alpha f(x).$$

That is, f is a solution of the differential equation

$$(1+x)h'(x) = \alpha h(x) \quad \text{for all } x \in (-1,1) \text{ and with } h(0) = 1. \tag{E}$$

We now show that f is actually the unique solution of this differential equation. Let k be another solution of (E) and define

$$h(x) = (1+x)^{-\alpha} k(x).$$

Note that h is well defined on $(-1, 1)$. Moreover, h is differentiable, and

$$\begin{aligned} h'(x) &= -\alpha(1+x)^{-\alpha-1}k(x) + (1+x)^{-\alpha}k'(x) \\ &= (1+x)^{-\alpha-1}\big(-\alpha k(x) + (1+x)k'(x)\big) = 0, \end{aligned}$$

where we use the assumption that k is a solution of (E). Hence, h is a constant on $(-1, 1)$. Since $h(0) = 1$, we have that $h(x) = 1$ for all x in $(-1, 1)$. That is,

$$k(x) = (1+x)^{\alpha} \quad \text{for all } x \in (-1,1).$$

This concludes the proof that there is a unique solution of (E).

We now need to show that the binomial power series is also a solution of (E). Let S be defined by

$$S(x) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k.$$

We first check that S is defined on $(-1, 1)$. For $k \geq 1$, let

$$a_k = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}|x^k|.$$

We have

$$\frac{|a_{k+1}|}{|a_k|} = \left|\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k)}{\alpha(\alpha-1)\cdots(\alpha-k+1)}\right| \frac{k!}{(k+1)!}|x| = \frac{|\alpha-k|}{k+1}|x|,$$

which converges to $|x|$ as k goes to infinity. By the ratio test the series converges for $|x| < 1$. Hence, S is defined on $(-1, 1)$. It turns out that power series are also differentiable on any open interval on which they are defined. Moreover, the derivative can be obtained by differentiating term by term. We will prove this result in Chap. 7. Hence,

$$\begin{aligned} S'(x) &= \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} k x^{k-1} \\ &= \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(k-1)!} x^{k-1}. \end{aligned}$$

We now compute $(1+x)S'(x) = S'(x) + xS'(x)$. We have

$$\begin{aligned}
S'(x) &= \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(k-1)!} x^{k-1} \\
&= \alpha + \sum_{k=2}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(k-1)!} x^{k-1} \\
&= \alpha + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k)}{k!} x^{k},
\end{aligned}$$

where the last equality comes from a shift of index. Therefore,

$$\begin{aligned}
S'(x) + xS'(x) = \alpha &+ \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k)}{k!} x^{k} \\
&+ \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(k-1)!} x^{k}.
\end{aligned}$$

Note that for all $k \geq 1$, we have

$$\begin{aligned}
&\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k)}{k!} + \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(k-1)!} \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(k-1)!}\left(1 + \frac{\alpha-k}{k}\right) = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\alpha.
\end{aligned}$$

Hence,

$$(1+x)S'(x) = \alpha + \alpha \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^{k} = \alpha S(x).$$

Moreover, $S(0) = 1$. Therefore, S is the unique solution of (E). This proves that $S(x) = (1+x)^{\alpha}$ and completes the proof of the binomial expansion.

Application 4.5 Consider the binomial series for $\alpha = n$ natural. We have

$$(1+x)^n = 1 + \sum_{k=1}^{\infty} a_k x^k$$

with, for $k \geq 1$,

$$a_k = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Note that

$$a_n = \frac{n(n-1)\cdots(n-n+1)}{n!} = 1$$

and that

$$a_{n+1} = \frac{n(n-1)\cdots(n-(n+1)+1)}{(n+1)!} = 0.$$

Moreover, for all $k > n$, we will have a factor $(n-n)$ in the numerator of a_k. Hence, $a_k = 0$ for all $k > n$. That is, the binomial series in the particular case of a natural exponent is a finite sum. We have

$$(1+x)^n = 1 + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k \quad \text{for all } x \in \mathbf{R}.$$

We will use a new notation to write this formula in a more convenient way. Note that by multiplying by $(n-k)!$ the numerator and denominator of a_k we get

$$\frac{n(n-1)\cdots(n-k+1)(n-k)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!}.$$

The binomial coefficients are defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for } 0 \le k \le n$$

that one reads: n choose k. Using the convention $0! = 1$, we get

$$\binom{n}{0} = 1.$$

Hence,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \text{for all } x \in \mathbf{R}.$$

This is a well-known algebraic formula. See the exercises for a more general form of this formula.

Application 4.6 Find the power series expansion of $(1+x)^{-1/2}$.

We use the binomial series expansion with $\alpha = -1/2$. We have

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{x^2}{2!} + \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\frac{x^3}{3!} + \cdots.$$

Define, for $\alpha = -1/2$,

$$b_k = \alpha(\alpha-1)\cdots(\alpha-k+1).$$

We now show by induction that

$$b_k = (-1)^k \frac{(2k)!}{2^{2k}k!} \quad \text{for all } k \ge 1.$$

For $k = 1$, the left-hand side is $b_1 = \alpha = -1/2$, and the right-hand side is

$$(-1)^1 \frac{2!}{2^2 1!} = \frac{-1}{2}.$$

Hence, the formula holds for $k = 1$. Assume that it holds for k. Then

$$b_{k+1} = \alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k) = (-1)^k \frac{(2k)!}{2^{2k}k!}(\alpha-k)$$

by the induction hypothesis. We have

$$b_{k+1} = (-1)^k \frac{(2k)!}{2^{2k}k!}\left(\frac{-1}{2} - k\right) = (-1)^k \frac{(2k)!}{2^{2k}k!}\frac{-2k-1}{2} = (-1)^{k+1}\frac{(2k+1)!}{2^{2k+1}k!}.$$

A little algebra yields

$$b_{k+1} = (-1)^{k+1}\frac{(2k+1)!}{2^{2k+1}k!}\frac{2k+2}{2k+2} = (-1)^{k+1}\frac{(2k+2)!}{2^{2k+2}(k+1)!},$$

where we have used in the denominator that

$$2^{2k+1}k!(2k+2) = 2^{2k+2}k!(k+1) = 2^{2k+2}(k+1)!.$$

This proves the formula by induction.

By the binomial series we have

$$(1+x)^{-1/2} = 1 + \sum_{k=1}^{\infty}\frac{b_k}{k!}x^k = 1 + \sum_{k=1}^{\infty}(-1)^k\frac{(2k)!}{2^{2k}(k!)^2}x^k$$

for all x in $(-1, 1)$.

Application 4.7 Find the power series expansion for arcsin. Recall that the derivative of arcsin is

$$(\arcsin)'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Substituting x by $-x^2$ into the power series expansion for $(1+x)^{-1/2}$, we get

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{\infty}(-1)^k\frac{(2k)!}{2^{2k}(k!)^2}\left(-x^2\right)^k = 1 + \sum_{k=1}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2}x^{2k}$$

for all x in $(-1, 1)$. It is possible to integrate a power series by integrating it term by term. This will be proved in Chap. 7. We use this property now. For y in $(-1, 1)$,

$$\int_0^y \frac{1}{\sqrt{1-x^2}}\,dy = \int_0^y 1\,dy + \sum_{k=1}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2}\int_0^y x^{2k}\,dy.$$

Since $\arcsin 0 = 0$, we have

$$\arcsin y = y + \sum_{k=1}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2}\frac{y^{2k+1}}{2k+1}$$

for all y in $(-1, 1)$.

Application 4.8 Use the power series expansion of arcsin to find an approximation of π.

Recall that $\arcsin(1/2) = \pi/6$. Hence,

$$\pi/6 = \arcsin(1/2) = 1/2 + \sum_{k=1}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2}\frac{(1/2)^{2k+1}}{2k+1}.$$

In particular, if we sum up to $k = 10$ (and multiply by 6), we get for π the approximation 3.141592647 for which the first seven decimals are correct.

Exercises

1. Prove that for any real y, we have
$$\frac{y^{2n+2}}{1+y^2} \leq y^{2n+2}.$$
2. Use the power series expansion of arctan at $x = \frac{1}{\sqrt{3}}$ to get the first 12 decimals of π.
3. (a) Show that $(-1)^{n+2}\frac{y^{n+2}}{n+2}$ is positive for $y < 0$.
   (b) Show that for fixed $y \leq 0$,
$$\lim_{n\to\infty}(-1)^{n+2}\frac{y^{n+2}}{n+2} = 0.$$
4. Assume that f and g have power series expansions
$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n$$
for x in [0, 1]. Let c and d be two reals. Show that $cf + dg$ also has a power series expansion on [0, 1].
5. Use the power series expansion of $\ln(1-x)$ to estimate $\ln 2$. Bound the error.
6. Show that
$$\ln(1-x) \leq -x$$
for all x in (0, 1).
7. Show that
$$\frac{1}{1-x} > 1 + x + x^2$$
for x in (0, 1).
8. If we approximate $\ln(1-x)$ by $-x - x^2/2$ for x in $(-1/2, 1/2)$, what is the maximum error we are making?
9. (a) The power series expansion of $\ln(1-x)$ is valid for x in $[-1, 1)$. Explain how it can be used to approximate $\ln y$ for any given $y > 0$.
   (b) Approximate $\ln 3$. Find a bound for the error.
10. The genius Ramanujan (1887–1920) proposed the following formula:
$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\frac{1103 + 26390n}{396^{4n}}.$$
Use it to estimate π.
11. Estimate Euler's constant (defined in Application 4.4). Bound the error.
12. Use Application 4.4 to show that
$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k}}{\ln n} = 1.$$
13. Recall that we say that a_n goes to positive infinity if for any A, there is a natural N such that $a_n > A$ for $n \geq N$. Assume that a_n and b_n go to infinity.

(a) Show that if $a_n - b_n$ converges to a finite limit, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

(b) Is the converse of (a) true?

14. Use the formula in Application 4.5 to show that for all reals a and b and naturals n, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

15. Use the binomial expansion to show that

$$1 - \sqrt{1-x} = \sum_{n=1}^{\infty} c_n x^n$$

for $|x| < 1$, where

$$c_n = \frac{(2n-2)!}{2^{2n-1} n!(n-1)!}.$$

4.2 Wallis' Integrals, Euler's Formula, and Stirling's Formula

For any integer $n \geq 0$, let

$$I_n = \int_0^{\pi/2} \sin^n x \, dx.$$

The integrals I_n are called Wallis' integrals. Observe that

$$I_0 = \int_0^{\pi/2} dx = \pi/2$$

and that

$$I_1 = \int_0^{\pi/2} \sin x \, dx = -\cos(\pi/2) + 1 = 1.$$

For $n \geq 2$, we use that $\sin^2 x = 1 - \cos^2 x$ to get

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^{n-2}(x)\big(1 - \cos^2 x\big) \, dx.$$

Hence,

$$I_n = I_{n-2} - \int_0^{\pi/2} \sin^{n-2}(x) \cos^2 x \, dx. \tag{4.2}$$

We now do a side computation. Recall from Calculus the integration-by-parts formula: if f and g are differentiable on an interval $[a, b]$ with continuous derivatives on the same interval, then

$$\int_a^b f'(x)g(x) \, dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x) \, dx.$$

Letting $f'(x) = \sin^{n-2}(x)\cos x$ and $f(x) = \frac{1}{n-1}\sin^{n-1}(x)$, $g(x) = \cos x$ and $g'(x) = -\sin x$, we get

$$\int_0^{\pi/2} \sin^{n-2}(x)\cos^2 x\,dx = \frac{1}{n-1}\big(\sin^{n-1}(\pi/2)\cos(\pi/2) - \sin^{n-1}(0)\cos(0)\big) - \int_0^{\pi/2} \frac{1}{n-1}\sin^{n-1}(x)(-\sin x)\,dx = \frac{I_n}{n-1}.$$

Plugging

$$\int_0^{\pi/2} \sin^{n-2}(x)\cos^2 x\,dx = \frac{I_n}{n-1}$$

into (4.2), we get

$$I_n = I_{n-2} - \frac{I_n}{n-1}.$$

That is,

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{for } n \geq 2.$$

As shown below, this recursion formula can be used to compute any I_n. Because the recursion formula links I_n to I_{n-2}, we have different formulas for even n and odd n.

Wallis' integrals

For any integer $n \geq 1$, we have

$$I_{2n+1} = \int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots(2n+1)}$$

and

$$I_{2n} = \int_0^{\pi/2} \sin^{2n} x\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\frac{\pi}{2}.$$

We prove by induction that

$$I_{2n+1} = \int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots(2n+1)}.$$

The proof for the other formula is similar and is left as an exercise. We have already computed $I_1 = 1$. Since

$$I_n = \frac{n-1}{n} I_{n-2},$$

we get

$$I_3 = \frac{2}{3} I_1 = \frac{2}{3}.$$

Therefore the formula above holds for $n = 1$. Assume that it holds for n. Then

$$I_{2(n+1)+1} = I_{2n+3} = \frac{2n+2}{2n+3} I_{2n+1} = \frac{2n+2}{2n+3} \cdot \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$
$$= \frac{2 \cdot 4 \cdot 6 \cdots 2n(2n+2)}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)}.$$

Thus, the formula holds for $n + 1$, and the formula is proved for all $n \geq 1$.

We now turn to the task of estimating π using Wallis' integrals. First note that

$$I_n = \int_0^{\pi/2} \sin^n x \, dx$$

is a decreasing sequence: $\sin x$ is positive on $[0, \pi/2]$ and is smaller than 1, thus

$$\sin^n x \leq \sin^{n-1} x \quad \text{for all } x \in [0, \pi/2],$$

and so

$$I_n = \int_0^{\pi/2} \sin^n x \, dx \leq \int_0^{\pi/2} \sin^{n-1}(x) \, dx = I_{n-1} \quad \text{for all } n \geq 1.$$

Hence, for $n \geq 1$,

$$\frac{I_n}{I_{n-1}} \leq 1.$$

On the other hand, we use the recursion formula and the fact that I_n is decreasing to get

$$I_n = \frac{n-1}{n} I_{n-2} \geq \frac{n-1}{n} I_{n-1}.$$

Therefore,

$$\frac{n-1}{n} \leq \frac{I_n}{I_{n-1}} \leq 1. \tag{4.3}$$

Thus, $\frac{I_n}{I_{n-1}}$ converges to 1 (why?). In particular,

$$\frac{I_{2n+1}}{I_{2n}}$$

converges to 1 (why?). We now use the explicit formulas for I_{2n} and I_{2n+1} to get

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots 2n-1} \frac{2}{\pi} = 1.$$

We rearrange the formula above to obtain

$$\lim_{n\to\infty} \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7} \cdots \frac{2n}{2n-1}\frac{2n}{2n+1} = \pi/2.$$

We introduce the following notation for products:

$$a_1 a_2 \cdots a_n = \prod_{i=1}^{n} a_i.$$

Note that

$$\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1}=\prod_{i=1}^{n}\frac{(2i)^2}{(2i-1)(2i+1)}=\prod_{i=1}^{n}\frac{4i^2}{4i^2-1}.$$

Hence,

$$\lim_{n\to\infty}\prod_{i=1}^{n}\frac{4i^2}{4i^2-1}=\frac{\pi}{2}.$$

We can now summarize our computations.

π as an infinite Wallis product
We have the following limit:

$$\lim_{n\to\infty}\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1}=\pi/2.$$

Alternatively, define the sequence p_n by

$$p_n=\prod_{i=1}^{n}\frac{4i^2}{4i^2-1}.$$

Then, the sequence p_n converges to $\pi/2$.

Note that for fixed n, p_n is a product with n factors, but in the limit (as n goes to infinity) $\pi/2$ is an infinite product.

Application 4.9 Estimate π using Wallis' infinite product. Using the formula above and keeping the first 9 decimals, one gets

$$p_{10}=1.533851903,\qquad p_{100}=1.566893745,\qquad p_{1000}=1.570403873.$$

The convergence to $\pi/2$ is very slow, p_{1000} has only the three first decimals of $\pi/2=1.5707963\ldots$.

We now bound the error of this method. We go back to Wallis' integrals and the definition of p_n to get

$$\frac{I_{2n+1}}{I_{2n}}=p_n\frac{2}{\pi}.$$

Using (4.3),

$$\frac{2n}{2n+1}\leq\frac{I_{2n+1}}{I_{2n}}\leq 1.$$

Thus,

$$\frac{2n}{2n+1}\frac{\pi}{2}\leq p_n\leq\frac{\pi}{2}.$$

But

$$\frac{2n}{2n+1} = \frac{2n+1}{2n+1} - \frac{1}{2n+1} = 1 - \frac{1}{2n+1},$$

so we have

$$0 \le \frac{\pi}{2} - p_n \le \frac{\pi}{2}\frac{1}{2n+1}.$$

In particular,

$$0 \le \frac{\pi}{2} - p_{1000} \le \frac{\pi}{2}\frac{1}{2001}.$$

While Wallis' products are not an efficient way to compute decimals for π, they are quite useful because they appear in a number of important formulas. The next application is a case in point.

Euler's formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

We follow the approach of Choe (1987) (American Mathematical Monthly, pp. 662–663). We start with the power series expansion for arcsin that we computed in Application 4.7 in Sect. 4.1:

$$\arcsin x = x + \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \frac{x^{2k+1}}{2k+1}.$$

It is easy to check that this power series has a radius of convergence of 1. We make the substitution $x = \sin t$, and we get

$$\arcsin(\sin t) = t = \sin t + \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{\sin^{2n+1} t}{2n+1}.$$

It turns out that we can integrate term by term the series on the right-hand side for t from 0 to $\pi/2$. We will use this fact without proving it, for a justification see Choe (1987). Hence,

$$\int_0^{\pi/2} t\,dt = \int_0^{\pi/2} \sin t\,dt + \sum_{k=n}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} \int_0^{\pi/2} \sin^{2n+1} t.$$

We recognize Wallis' integral

$$I_{2n+1} = \int_0^{\pi/2} \sin^{2n+1} t = \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots (2n+1)}.$$

Note first that

$$2 \cdot 4 \cdot 6 \cdots (2n) = 2^n (1 \cdot 2 \cdot 3 \cdots n) = 2^n n!. \tag{4.4}$$

Observe also that

$$1 \cdot 3 \cdot 5 \cdots (2n+1) \times 2 \cdot 4 \cdot 6 \cdots (2n) = (2n+1)!. \tag{4.5}$$

By multiplying the numerator and denominator by $2 \cdot 4 \cdot 6 \cdots (2n)$ we get

$$I_{2n+1} = \frac{(2 \cdot 4 \cdot 6 \cdots 2n)^2}{(2 \cdot 4 \cdots 2n) 3 \cdot 5 \cdot 7 \cdots (2n+1)} = \frac{(2^n n!)^2}{(2n+1)!},$$

where we used (4.4) in the numerator and (4.5) in the denominator. Substituting I_{2n+1} by the expression we just found, we get

$$\int_0^{\pi/2} t\,dt = \int_0^{\pi/2} \sin t\,dt + \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} \frac{(2^n n!)^2}{(2n+1)!}.$$

We now need three simple computations:

$$\int_0^{\pi/2} t\,dt = t^2/2\Big]_0^{\pi/2} = \pi^2/8,$$

$$\int_0^{\pi/2} \sin t\,dt = -\cos t\Big]_0^{\pi/2} = 1,$$

and

$$\frac{(2n)!}{2^{2n}(n!)^2(2n+1)} \frac{(2^n n!)^2}{(2n+1)!} = \frac{1}{(2n+1)^2}.$$

Using these three computations in the series above, we have

$$\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}.$$

We now need a short step to get Euler's formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}.$$

But

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solving for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ yields

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We now turn to Stirling's formula. It provides a very useful estimate for $n!$ when n is large.

Stirling's formula

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}\, e^{-n} n^{n+1/2}} = 1.$$

In other words, as n gets large, $n!$ can be approximated by the function $\sqrt{2\pi}\, e^{-n} n^{n+1/2}$. Before proving the formula, we will explain how to come up with an estimate for $n!$. Consider first

$$\ln n! = \sum_{k=1}^{n} \ln k = \sum_{k=2}^{n} \ln k.$$

Using that ln is an increasing function, we get

$$\ln x < \ln k \quad \text{for } x \text{ in } (k-1, k).$$

We integrate x from $k-1$ to k to get

$$\int_{k-1}^{k} \ln x\, dx \le \int_{k-1}^{k} \ln k\, dx = \ln k.$$

Using that

$$\ln k < \ln x \quad \text{for } x \text{ in } (k, k+1)$$

and integrating for x between k and $k+1$, we get

$$\ln k < \int_{k}^{k+1} \ln x\, dx.$$

Therefore,

$$\int_{k-1}^{k} \ln x\, dx \le \ln k \le \int_{k}^{k+1} \ln x\, dx.$$

We now sum from $k = 2$ to n to get

$$\sum_{k=2}^{n} \int_{k-1}^{k} \ln x\, dx \le \sum_{k=2}^{n} \ln k \le \sum_{k=2}^{n} \int_{k}^{k+1} \ln x\, dx.$$

But

$$\sum_{k=2}^{n} \int_{k-1}^{k} \ln x\, dx = \int_{1}^{2} \ln x\, dx + \int_{2}^{3} \ln x\, dx + \cdots + \int_{n-1}^{n} \ln x\, dx = \int_{1}^{n} \ln x\, dx.$$

Hence,

$$\int_1^n \ln x\, dx \le \sum_{k=2}^{n} \ln k \le \int_2^{n+1} \ln x\, dx.$$

Using that an antiderivative for $\ln x$ is $x \ln x - x$, we get

$$n \ln n - n - 1 \ln 1 = n \ln n - n \le \ln n! \le (n+1)\ln(n+1) - (n+1) - 2\ln 2.$$

That is, $\ln n!$ is of the same order as $n \ln n - n$.

We now prove Stirling's formula. The sequence $(n+1/2)\ln n - n$ is between the two bounds of $\ln n!$ we computed above. This motivates the definition of

$$S_n = \ln n! - (n+1/2)\ln n + n.$$

The main task is to prove that S_n converges. In order to do so, we will show that S_n is the partial sum of a convergent series. The general term of the series is defined by

$$\begin{aligned} u_n = S_n - S_{n-1} = \ln\left(\frac{n!}{(n-1)!}\right) &- (n+1/2)\ln n + n \\ &+ (n-1/2)\ln(n-1) - (n-1). \end{aligned}$$

A little algebra yields

$$u_n = 1 + (n-1/2)\ln\left(\frac{n-1}{n}\right).$$

Recall the power series expansion

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{for all } x \in [-1, 1).$$

Letting $x = 1/n$, we get, for $n \ge 2$,

$$\ln\left(\frac{n-1}{n}\right) = \ln(1-1/n) = -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} - \cdots.$$

Hence, by omitting all the terms (they are all negative) but the first two we get

$$\ln(1-1/n) < -\frac{1}{n} - \frac{1}{2n^2}$$

and

$$u_n < 1 + (n-1/2)\left(-\frac{1}{n} - \frac{1}{2n^2}\right) = \frac{1}{4n^2}.$$

On the other hand, using that

$$\ln(1-1/n) = -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} - \frac{1}{4n^4} - \cdots > -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{2n^3} - \frac{1}{2n^4} - \cdots,$$

we get

$$\ln(1-1/n) > -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{2n^3} - \frac{1}{2n^4} - \cdots$$
$$= -\frac{1}{n} - \frac{1}{2n^2}\left(1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots\right).$$

Summing the geometric series with ratio $r = 1/n$, we get

$$\ln(1-1/n) > -\frac{1}{n} - \frac{1}{2n^2}\frac{1}{1-1/n}.$$

Hence,

$$u_n > 1 + (n-1/2)\left(-\frac{1}{n} - \frac{1}{2n(n-1)}\right) = -\frac{1}{4(n-1)n}.$$

Therefore, we have, for all $n \geq 2$,

$$-\frac{1}{4(n-1)n} < u_n < \frac{1}{4n^2}.$$

Thus,

$$|u_n| < \frac{1}{4n(n-1)}$$

(why?). It is clear that

$$\lim_{n\to\infty} \frac{\frac{1}{4n(n-1)}}{\frac{1}{4n^2}} = 1.$$

Hence, by the limit comparison test the series $\sum_{n=1}^{\infty} \frac{1}{4n(n-1)}$ converges, and by the comparison test the series $\sum_{n=1}^{\infty} u_n$ converges absolutely. Hence, the partial sum S_n converges to some limit ℓ. That is,

$$\lim_{n\to\infty} \big(\ln n! - (n+1/2)\ln n + n\big) = \ell.$$

Since exp is a continuous function on the reals, we get

$$\lim_{n\to\infty} \exp\big(\ln n! - (n+1/2)\ln n + n\big) = \exp(\ell),$$

and so

$$\lim_{n\to\infty} n! n^{-n-1/2}\exp(n) = \exp(\ell).$$

Set $C = \exp(\ell)$. We have

$$\lim_{n\to\infty} \frac{n!}{Cn^{n+1/2}\exp(-n)} = 1.$$

The only remaining task to prove Stirling's formula is to compute C. In order to do so, we use Wallis' formula. Recall that

$$\lim_{n\to\infty} \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1} = \pi/2.$$

By (4.4)

$$2 \cdot 2 \cdot 4 \cdot 4 \cdots (2n)(2n) = \big(2 \cdot 4 \cdot 6 \cdots (2n)\big)^2 = \big(2^n n!\big)^2 = 2^{2n}(n!)^2.$$

By (4.5)

$$1 \cdot 3 \cdot 3 \cdots (2n-1)(2n-1)(2n+1) \times 2 \cdot 2 \cdot 4 \cdot 4 \cdots (2n)(2n) = \big((2n)!\big)^2 (2n+1).$$

Therefore,

$$1 \cdot 3 \cdot 3 \cdots (2n-1)(2n-1)(2n+1) \times 2^{2n}(n!)^2 = \big((2n)!\big)^2 (2n+1).$$

We multiply the numerator and the denominator of

$$\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1}$$

by $2^{2n}(n!)^2$ to get

$$\frac{2^{2n}(n!)^2}{2^{2n}(n!)^2}\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1} = \frac{(2^{2n}(n!)^2)^2}{((2n)!)^2(2n+1)}.$$

An alternative to this is to do an induction proof to show that

$$\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1} = \frac{(2^{2n}(n!)^2)^2}{((2n)!)^2(2n+1)}$$

for all $n \geq 1$. See the exercises.

In any case, Wallis' formula becomes

$$\lim_{n\to\infty} \frac{(2^{2n}(n!)^2)^2}{((2n)!)^2(2n+1)} = \frac{\pi}{2}.$$

We now substitute every factorial $k!$ by its equivalent

$$Ck^{k+1/2}\exp(-k)$$

in the limit above (why can we do that?). We have

$$\lim_{n\to\infty} \frac{2^{4n}C^4 e^{-4n} n^{4n+2}}{(Ce^{-2n}(2n)^{2n+1/2})^2(2n+1)} = \frac{\pi}{2}.$$

After simplification we get

$$\lim_{n\to\infty} C^2 \frac{n}{2(2n+1)} = \frac{\pi}{2}.$$

Thus,

$$C^2/4 = \pi/2.$$

That is, $C = \sqrt{2\pi}$. This completes the proof of Stirling's formula.

Exercises

1. Prove that for any integer $n \geq 1$, we have
$$I_{2n} = \int_0^{\pi/2} \sin^{2n} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2}.$$
2. Show that if the sequence a_n is such that a_n/a_{n-1} converges to 1, then a_{2n+1}/a_{2n} converges.
3. Assume that the sequence a_n does not take the 0 value and that it converges to $\ell \neq 0$.
   (a) Show that a_n/a_{n-1} converges to 1.
   (b) Is (a) still true for $\ell = 0$?
4. (a) Find a sequence of rational numbers that converge to π.
   (b) Does this mean that π is rational?
5. (a) Find the first four decimals of π using Wallis' method.
   (b) How many terms of the sequence p_n do you need to compute to get six decimals for π?
6. Show that the sequence p_n (from Wallis' method) is increasing.
7. Assume that a_n is a sequence and assume that there is an N and an M so that
$$|a_n| < M \quad \text{for } n \geq N.$$
   Show that a_n is bounded.
8. Prove by induction that
$$\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1} = \frac{(2^{2n}(n!)^2)^2}{((2n)!)^2(2n+1)}.$$
9. In this exercise we use infinite series and Wallis' products to estimate π. The method is due to Euler.
   Our starting point is
$$\pi/4 = \int_0^1 \frac{1}{1+y^2}\, dy.$$
   (a) Make the change of variable $y = \sqrt{1-s}$ in the integral above to get
$$\pi/4 = \int_0^1 \frac{1}{2-s} \frac{1}{\sqrt{1-s}}\, ds.$$
   (b) Show that for s in [0,1], we have
$$\frac{1}{2-s} = \sum_{n=0}^{\infty} \frac{s^n}{2^{n+1}}.$$
   (c) Use (c) in (b) to get
$$\pi/4 = \sum_{n=0}^{\infty} \int_0^1 \frac{s^n}{2^{n+1}} \frac{1}{\sqrt{1-s}}\, ds.$$

(In order to get this formula, one must interchange the sign $\int_0^1$ with the sign $\sum_{n=0}^{\infty}$. This is not always valid, but in this case it is; we will not prove this step.)

(d) Make the change of variable $s = \sin^2 t$ to show that

$$\int_0^1 \frac{s^n}{2} \frac{1}{\sqrt{1-s}}\, ds = \int_0^{\pi/2} \sin^{2n+1} t\, dt.$$

(e) Observe that the integral in (d) is a Wallis' integral. Show that

$$\int_0^{\pi/2} \sin^{2n+1} t\, dt = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

(f) Show that

$$\pi/4 = \sum_{n=0}^{\infty} \frac{2^n (n!)^2}{(2n+1)!}.$$

(g) Use (f) to estimate π.

10. Let

$$d_k = C k^{k+1/2} \exp(-k),$$

where C is a strictly positive constant. At some point in the proof of Stirling's formula, we have that

$$\lim_{k\to\infty} \frac{k!}{d_k} = 1$$

and that

$$\lim_{n\to\infty} \frac{(2^{2n}(n!)^2)^2}{((2n)!)^2(2n+1)} = \frac{\pi}{2}.$$

Show that

$$\lim_{n\to\infty} \frac{(2^{2n}(d_n)^2)^2}{(d_{2n})^2(2n+1)} = \frac{\pi}{2}.$$

11. Compute

$$\frac{n!}{\sqrt{2\pi}\, e^{-n} n^{n+1/2}}$$

for $n = 10, 100,$ and 1000.

12. Find an estimate (that does not involve a factorial) for

$$\frac{n!}{n^n}$$

when n is large.

13. (a) Show that the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{1}{2}\right)^{2n}$$

diverges.

(b) Take p in $(0, 1)$. Show that the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} p^n (1-p)^n$$

converges for all $p \neq 1/2$.

14. In this exercise we show that Stirling's estimate is actually a lower bound of $n!$. Let

$$u_n = 1 + (n - 1/2) \ln\left(\frac{n-1}{n}\right)$$

and $S_n = \sum_{k=2}^{n} u_k$.

(a) Show that for $n \geq 2$, we have

$$\ln(1 - 1/n) < -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3}.$$

(b) Use (a) to show that $u_n < 0$ for all $n \geq 2$.
(c) Show that S_n is a decreasing sequence.
(d) Conclude that for all $n \geq 2$,

$$n! > \sqrt{2\pi} e^{-n} n^{n+1/2}.$$

4.3 Convergence of Infinite Products

In this section we are interested in the convergence of sequences of the type

$$q_n = \prod_{i=1}^{n} a_i.$$

In the preceding section Wallis' formula is such an infinite product. As the reader will see, there is a close connection to infinite series. We first show that the only interesting infinite products are the ones whose general term converges to 1.

Example 4.1 Let a_n be a sequence such that $a_n \neq 0$ for all $n \geq 1$. Let $q_n = \prod_{i=1}^{n} a_i$. Assume that q_n converges to a nonzero limit ℓ. This implies that a_n converges to 1.

The proof is very easy. Note that

$$a_n = \frac{q_n}{q_{n-1}}.$$

Since q_n converges to $\ell \neq 0$, we have that

$$\frac{q_n}{q_{n-1}}$$

converges to 1 (why?). Thus, a_n converges to 1, and we are done.

We are now going to find necessary and sufficient conditions for the convergence of infinite products in two particular cases. First, we consider the case where the general term of the product is larger than 1.

Convergence of $\prod_{i=1}^{\infty}(1+u_i)$
Assume that u_i is a sequence of positive numbers. The sequence

$$q_n = \prod_{i=1}^{n}(1+u_i)$$

converges if and only if the series

$$\sum_{i=1}^{\infty} u_i$$

converges.

At first glance the reader may be surprised that an infinite product of reals strictly larger than 1 may converge to a finite limit. What the result above says is that convergence happens if only if the general term of the product converges to 1 fast enough.

Assume first that the infinite series $\sum_{i=1}^{\infty} u_i$ converges. Recall that

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \geq 1+x$$

for $x \geq 0$. Thus,

$$q_n = \prod_{i=1}^{n}(1+u_i) \leq \prod_{i=1}^{n}\exp(u_i) = \exp\left(\sum_{i=1}^{n} u_i\right) \leq \exp\left(\sum_{i=1}^{\infty} u_i\right) = C,$$

where the last inequality comes from the facts that the sequence u_i is assumed to be positive and exp is an increasing function. Note also that since $\sum_{i=1}^{\infty} u_i$ converges, the constant C is finite. On the other hand,

$$q_{n+1} = (1+u_{n+1})q_n \geq q_n.$$

That is, q_n is an increasing sequence. Thus, q_n is increasing and bounded, and thus it must converge. We have proved one implication.

For the converse, assume that q_n converges. Let

$$S_n = \sum_{i=1}^{n} u_i.$$

Then S_n is increasing since $u_i \geq 0$. Note that

$$S_n = 1 + u_1 + \cdots + u_n \leq (1+u_1)(1+u_2)\cdots(1+u_n) = \prod_{i=1}^{n}(1+u_i) = q_n.$$

Since q_n converges, it must be bounded. Therefore, S_n is also bounded, and since it is increasing, it converges. That is, the infinite series $\sum_{i=1}^{\infty} u_i$ converges, and the proof is complete.

Example 4.2 Consider the Wallis' product $\prod_{i=1}^{n} \frac{4i^2}{4i^2-1}$. We already know that it converges to $\pi/2$. If we did not know that, can we apply the criterion above to check that it converges?

Note that

$$\frac{4i^2}{4i^2-1} = \frac{4i^2-1}{4i^2-1} + \frac{1}{4i^2-1} = 1 + \frac{1}{4i^2-1} = 1 + u_i,$$

where

$$u_i = \frac{1}{4i^2-1} > 0.$$

The criterion above may be applied since the general term of the product is of the type $1 + u_i$ with $u_i \geq 0$. Note that

$$\lim_{n\to\infty} \frac{\frac{1}{4i^2-1}}{\frac{1}{4i^2}} = 1.$$

By the p test the series $\sum_{i=1}^{\infty} \frac{1}{4i^2}$ converges. By the limit comparison test $\sum_{i=1}^{\infty} \frac{1}{4i^2-1}$ converges as well. According to our criterion,

$$\prod_{i=1}^{n} \frac{4i^2}{4i^2-1}$$

converges, and we are done.

We now turn to another particular case of product. This time we assume that the general term is less than 1.

Convergence of $\prod_{i=1}^{\infty}(1-u_i)$

Assume that u_i is a sequence of numbers in $[0, 1)$. The sequence

$$q_n = \prod_{i=1}^{n}(1-u_i)$$

converges to a strictly positive limit if and only if the series

$$\sum_{i=1}^{\infty} u_i$$

converges.

It may be surprising that an infinite product of numbers strictly less than 1 converges to a nonzero limit. Similarly to the result above, this happens if and only if the general term of the product converges to 1 fast enough.

We first show that q_n converges. Note that $q_n > 0$ and that

$$q_{n+1} = (1 - u_{n+1})q_n \le q_n.$$

Thus, q_n is decreasing and bounded below by 0, and thus it must converge to some limit ℓ. Moreover, since $q_n \le 1$ for all $n \ge 1$, we must have $\ell \le 1$. Is $\ell = 0$? This is a more delicate point.

Assume first that the series $\sum_{i=1}^{\infty} u_i$ converges. Define the sequence

$$S_n = \sum_{i=1}^{n} u_i.$$

By the definition of convergence of the series $\sum_{i=1}^{\infty} u_i$, the sequence S_n converges to the limit $S = \sum_{i=1}^{\infty} u_i$. Set $\epsilon = 1/2$. There is N such that if $n \ge N$, then

$$|S_n - S| < 1/2.$$

Thus,

$$\left| \sum_{i=N+1}^{\infty} u_i \right| = \sum_{i=N+1}^{\infty} u_i < 1/2.$$

We now show, for all naturals k, that

$$\prod_{i=N+1}^{N+k} (1 - u_i) \ge 1 - \sum_{i=N+1}^{N+k} u_i.$$

We first verify that the inequality holds for $k = 1$:

$$1 - u_{N+1} = 1 - u_{N+1}.$$

Therefore, the inequality holds for $k = 1$. Assume that it holds for k. Then,

$$\prod_{i=N+1}^{N+k+1} (1 - u_i) = (1 - u_{N+k+1}) \prod_{i=N+1}^{N+k} (1 - u_i) \ge (1 - u_{N+k+1}) \left(1 - \sum_{i=N+1}^{N+k} u_i \right).$$

Expanding the last term yields

$$\begin{aligned}(1 - u_{N+k+1}) \left(1 - \sum_{i=N+1}^{N+k} u_i \right) &= 1 - \sum_{i=N+1}^{N+k} u_i - u_{N+k+1} + u_{N+k+1} \sum_{i=N+1}^{N+k} u_i \\ &\ge 1 - \sum_{i=N+1}^{N+k+1} u_i.\end{aligned}$$

This proves the inequality by induction. Putting together the inequality just proved and that

$$\sum_{i=N+1}^{\infty} u_i < 1/2,$$

we get, for all $k \geq 1$,

$$\prod_{i=N+1}^{N+k} (1-u_i) \geq 1 - \sum_{i=N+1}^{N+k} u_i \geq 1 - \sum_{i=N+1}^{\infty} u_i > 1 - 1/2 = 1/2.$$

That is, for all $n > N$, we have

$$\prod_{i=N+1}^{n} (1-u_i) > 1/2.$$

Observe now that, for $n > N$,

$$q_n = q_N \prod_{i=N+1}^{n} (1-u_i) > \frac{1}{2} q_N.$$

In particular, the limit ℓ is larger than $\frac{1}{2} q_N > 0$. This proves that $\ell > 0$.

We now deal with the converse. Assume that $\ell > 0$. There exists an integer K such that if $k \geq K$, then

$$|q_k - \ell| < \ell/2.$$

In particular,

$$q_k > \ell/2 \quad \text{for } k \geq K.$$

On the other hand, it is easy to show that

$$1 - x \leq e^{-x} \quad \text{for all } x \geq 0.$$

For if g is defined by $g(x) = \exp(-x) - (1-x)$, then g is differentiable, and $g'(x) = -\exp(-x) + 1 \geq 0$ for $x \geq 0$ (why?). Thus, g is increasing on $[0, \infty)$, and so $g(x) \geq g(0) = 0$ for all $x \geq 0$. This proves the inequality. Therefore,

$$q_k = \prod_{i=1}^{k} (1-u_i) \leq \prod_{i=1}^{k} \exp(-u_i) = \exp\left(-\sum_{i=1}^{k} u_i\right) = \exp(-S_k).$$

Since $q_k > \ell/2$ for $k \geq K$, we have

$$\ln\big(\exp(-S_k)\big) \geq \ln(q_k) > \ln(\ell/2) \quad \text{for } k \geq K.$$

Hence,

$$S_k < -\ln(\ell/2) \quad \text{for } k \geq K.$$

Therefore, S_n is bounded above (why?) and increasing (why?). Thus, S_n converges, and we are done.

Example 4.3 Does the sequence

$$q_n = \prod_{i=1}^{n} \cos\big(\pi/2^{i+1}\big)$$

converge? We have

$$\cos\left(\pi/2^{i+1}\right) = 1 - u_i$$

with $u_i = 1 - \cos(\pi/2^{i+1}) \geq 0$. Therefore, this sequence converges. Does it converge to a strictly positive limit?

Recall that

$$\cos x \geq 1 - x^2/2 \quad \text{for all } x.$$

Thus,

$$0 \leq u_i \leq \left(\pi/2^{i+1}\right)^2/2 = \frac{\pi^2}{2^{2i+3}}.$$

The series

$$\sum_{i=1}^{\infty} \frac{\pi^2}{2^{2i+3}} = \frac{\pi^2}{8}\frac{1}{3}$$

converges. Thus, the series $\sum_{i=1}^{\infty} u_i$ converges as well (why?). Therefore, the infinite product

$$\prod_{i=1}^{n} \cos\left(\pi/2^{i+1}\right)$$

converges to a strictly positive limit. In fact, in the exercises it is shown that the limit is $2/\pi$.

At this point we have two necessary and sufficient conditions for convergence of infinite products. However, these results apply to particular infinite products (the general term must be always above 1 or always below 1). We now turn to a condition for convergence that applies to any infinite product but is only a sufficient condition.

A sufficient condition for convergence
Let u_i be a sequence of real numbers in $(-1, +\infty)$. If the sequence

$$p_n = \prod_{i=1}^{n} \left(1 + |u_i|\right)$$

converges, then the sequence

$$q_n = \prod_{i=1}^{n} (1 + u_i)$$

converges to a strictly positive limit.

The result above is closely related to the corresponding result for series: absolute convergence implies convergence.

We first need two simple lemmas.

Lemma 4.1 *For x in $[0, 1)$, we have*

$$\ln(1+x) \le \frac{x}{1-x}.$$

To prove Lemma 4.1, we start with the power series expansion

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} x^n/n = x - x^2/2 + x^3/3 + \cdots,$$

which holds for all x in $[-1, 1)$. Observe now that for $x \ge 0$ and $n \ge 1$, we have

$$(-1)^{n+1} x^n/n \le x^n/n \le x^n.$$

Thus,

$$\ln(1+x) \le \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \quad \text{for all } x \in [0, 1).$$

This proves Lemma 4.1.

Lemma 4.2 *For all y in $[-1/4, 1/4]$, we have*

$$\left|\ln(1+y)\right| \le 2|y|.$$

If $y \ge 0$, we have, by Lemma 4.1,

$$\left|\ln(1+y)\right| = \ln(1+y) \le \frac{y}{1-y}.$$

If $y \le 1/4$, then $1 - y \ge 1 - 1/4$ and

$$\frac{1}{1-y} \le 4/3 < 2.$$

Hence,

$$\frac{y}{1-y} \le 4y/3 < 2y$$

and

$$\left|\ln(1+y)\right| \le 2y = 2|y| \quad \text{for all } y \in [0, 1/4].$$

We now turn to y in $[-1/4, 0)$. Since $1 + y < 1$, $\ln(1 + y) < 0$. Therefore,

$$\left|\ln(1+y)\right| = -\ln(1+y) = \ln\left(\frac{1}{1+y}\right) = \ln\left(1 - \frac{y}{1+y}\right).$$

Using that $y < 0$, we get

$$\frac{y}{1+y} = \frac{-|y|}{1-|y|}.$$

Hence,

$$\left|\ln(1+y)\right| = \ln\left(1 + \frac{|y|}{1-|y|}\right).$$

Note that if $|y| < 1/2$, then

$$\frac{|y|}{1-|y|} < 1,$$

and we may apply Lemma 4.1 to

$$x = \frac{|y|}{1-|y|}$$

to get

$$\left|\ln(1+y)\right| = \ln\left(1+\frac{|y|}{1-|y|}\right) = \ln(1+x) \le \frac{x}{1-x} = \frac{|y|}{1-2|y|}.$$

But

$$\frac{1}{1-2|y|} \le 2 \quad \text{for } |y| \le 1/4.$$

Therefore,

$$\left|\ln(1+y)\right| \le 2|y|.$$

This completes the proof of Lemma 4.2.

We are now ready to show that absolute convergence implies convergence for infinite products. Assume that

$$p_n = \prod_{i=1}^{n}\left(1+|u_i|\right)$$

converges. Since $|u_i| \ge 0$, we know that if p_n converges, then

$$\sum_{i=1}^{\infty} |u_i|$$

converges. In particular, $|u_i|$ converges to 0, and there exists N such that if $i \ge N$, then

$$|u_i| < 1/4.$$

Hence, by Lemma 4.2,

$$\left|\ln(1+u_i)\right| < 2|u_i| \quad \text{for } i \ge N.$$

By the series comparison test the positive terms series

$$\sum_{i=1}^{\infty}\left|\ln(1+u_i)\right|$$

converges. Hence, the series

$$\sum_{i=1}^{\infty} \ln(1+u_i)$$

converges absolutely and therefore converges. Let

$$S_n = \sum_{i=1}^{n} \ln(1 + u_i).$$

The sequence S_n converges to some limit S. By the continuity of the exp function, $\exp(S_n)$ converges to $\exp(S)$. But

$$\exp(S_n) = \prod_{i=1}^{n}(1 + u_i) = q_n.$$

This proves that q_n converges to $\exp(S) > 0$. We have proved that absolute convergence implies convergence for infinite products.

Example 4.4 Does the infinite product

$$\prod_{i=2}^{n}\left(1 + \frac{(-1)^n}{n^2}\right)$$

converge?

Since

$$\sum_{i=2}^{n}\left|\frac{(-1)^n}{n^2}\right|$$

converges, so does

$$\prod_{i=2}^{n}\left(1 + \left|\frac{(-1)^n}{n^2}\right|\right),$$

and therefore,

$$\prod_{i=2}^{n}\left(1 + \frac{(-1)^n}{n^2}\right)$$

converges absolutely, and so it converges.

Example 4.5 Does the infinite product

$$\prod_{i=2}^{n}\left(1 + \frac{(-1)^n}{n}\right)$$

converge?

In this case the series

$$\sum_{i=2}^{n}\left|\frac{(-1)^n}{n}\right|$$

diverges, so

$$\prod_{i=2}^{n}\left(1+\left|\frac{(-1)^n}{n}\right|\right)$$

diverges as well. This shows that

$$\prod_{i=2}^{n}\left(1+\frac{(-1)^n}{n}\right)$$

does not converge absolutely. This does not prove that it diverges. It actually converges, but we need another criterion to prove it. See the exercises.

Exercises

1. Consider the sequence

$$q_n=\prod_{i=1}^{n}\left(1+\frac{1}{i^p}\right).$$

   Find the values of p for which q_n converges.
2. Find the limit of

$$\prod_{i=2}^{n}\left(1-\frac{1}{i}\right).$$

3. Find the values of p for which

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^p}\right)$$

   converges to a strictly positive limit.
4. Assume that $|x|<1$.
   (a) Show that

$$\left|\ln(1+x)-x\right|<\frac{x^2}{2(1-|x|)}.$$

   (b) Use (a) to show that if $|x|<1/2$, then

$$\left|\ln(1+x)-x\right|<x^2.$$

5. In this exercise we prove another necessary and sufficient condition for convergence for infinite products. Consider $\prod_{n=1}^{\infty}(1+a_n)$ with all $a_n>-1$ and such that

$$\sum_{n=1}^{\infty}a_n^2 \text{ converges.}$$

   Under this condition, we show that $\prod_{n=1}^{\infty}(1+a_n)$ converges to a nonzero limit if and only if $\sum_{n=1}^{\infty}a_n$ converges.

(a) Show there is a natural N such that if $n \geq N$, then

$$|a_n| < 1/2.$$

(b) Use Exercise 4(b) to show that for $n \geq N$, we have

$$\left|\ln(1+a_n) - a_n\right| < a_n^2.$$

(c) Show that the series

$$\sum_{n=1}^{\infty} \left(\ln(1+a_n) - a_n\right)$$

converges.

(d) Let

$$p_n = \prod_{i=1}^{n} (1+a_i) \quad \text{and} \quad S_n = \sum_{i=1}^{n} a_i.$$

Use (c) to show that the sequence $\ln p_n - S_n$ converges.

(e) Conclude that $\prod_{n=1}^{\infty}(1+a_n)$ converges to a nonzero limit if and only if $\sum_{n=1}^{\infty} a_n$ converges.

6. Apply the result in Exercise 5 to show that

$$\prod_{n=1}^{\infty} \left(1 + \frac{(-1)^n}{n}\right)$$

converges.

7. In this exercise we describe another infinite product that may be used to estimate π. This method is due to Viète. Let $\alpha \neq 0$.

(a) Show that for every natural n, we have

$$\frac{\sin \alpha}{\alpha} = \cos\frac{\alpha}{2} \cos\frac{\alpha}{2^2} \cos\frac{\alpha}{2^2} \cdots \cos\frac{\alpha}{2^n} \frac{\sin(\frac{\alpha}{2^n})}{\frac{\alpha}{2^n}}.$$

(Use that $\sin(2x) = 2\cos x \sin x$.)

(b) Show that

$$\lim_{n\to\infty} \frac{\sin(\frac{\alpha}{2^n})}{\frac{\alpha}{2^n}} = \cos 0.$$

(c) Let

$$q_n = \cos\frac{\alpha}{2} \cos\frac{\alpha}{2^2} \cos\frac{\alpha}{2^2} \cdots \cos\frac{\alpha}{2^n}.$$

Show that q_n converges to $\frac{\sin\alpha}{\alpha}$.

(d) Let $a_n = \cos(\pi/2^{n+1})$. Show that

$$\lim_{n\to\infty} \prod_{i=1}^{n} a_i = \frac{2}{\pi}.$$

(Use (c) for a suitable value of α.)

(e) Show that $a_1 = \sqrt{2}/2$ and that, for $n \geq 1$,

$$a_{n+1} = \sqrt{\frac{1}{2}(1 + a_n)}.$$

(Prove and use that $\cos^2 x = \frac{1+\cos(2x)}{2}$.)

(f) Use (d) and (e) to estimate π.

4.4 The Number π Is Irrational

All the methods we have seen so far to estimate π involve infinity: infinite products and infinite series. Is it possible to avoid that? If π were a rational, for instance, it would be a fraction. No need for infinite series or infinite products! However, we will show that π is not a rational. Worse than that, it is known that π is not even an algebraic number. That is, there is no polynomial with integer coefficients that has π as a root. Recall that $\sqrt{2}$ is not a rational. However, it is algebraic: it is a solution of $x^2 - 2 = 0$. This allows simple methods to estimate $\sqrt{2}$: $1.4^2 = 1.96$ and $1.5^2 = 2.25$, and thus,

$$1.4 < \sqrt{2} < 1.5.$$

Since $(1.45)^2 > 2$, we have

$$1.4 < \sqrt{2} < 1.45,$$

and so on. We may get an arbitrarily precise estimate for $\sqrt{2}$ just using algebraic methods (additions and multiplications). A number, such as π, which is not algebraic is called transcendental (the number e is another example). As Euler said "it transcends the power of algebraic methods". One cannot avoid infinity to estimate π or e.

To prove that π is transcendental is rather involved. We will prove the weaker result that

The number π is irrational.

This already requires several steps. We follow the proof of Ivan Niven (1947) (Bulletin of the American Mathematical Society, 509). Assume, by contradiction, that π is a rational a/b where a and b are natural numbers. Pick a natural n such that

$$\frac{\pi^{n+1} a^n}{n!} < 1.$$

(Why does such an n exist?) Let f_n be the polynomial defined by

$$f_n(x) = \frac{x^n (a - bx)^n}{n!}.$$

Hence, $n!f_n(x) = x^n(a - bx)^n$. Since a and b are integers, if we expand $x^n(a - bx)^n$, all the coefficients are going to be integers. Moreover, all the powers in the expansion are going to be between n and $2n$. More precisely, there are integers $c_n, c_{n+1}, \ldots, c_{2n}$ such that

$$n!f_n(x) = \sum_{k=n}^{2n} c_k x^k.$$

At this point we need the following:

Lemma 4.3 *Assume that P is a polynomial of degree K. That is, there are $K + 1$ real numbers c_k such that*

$$P(x) = \sum_{k=0}^{K} c_k x^k.$$

Let the jth derivative of P be $P^{(j)}$. Then,

$$P^{(j)}(0) = j!c_j \quad \text{for } 1 \le j \le K,$$

and

$$P^{(j)}(x) = 0 \quad \text{for } j > K.$$

We prove Lemma 4.3. We have

$$P(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_k x^k + \cdots + c_K x^K.$$

We differentiate to get

$$P'(x) = P^{(1)}(x) = c_1 + 2c_2 x + \cdots + kc_k x^{k-1} + \cdots + Kc_K x^{K-1}.$$

We differentiate again to get

$$P^{(2)}(x) = 2c_2 + \cdots + k(k-1)c_k x^{k-2} + \cdots + K(K-1)c_K x^{K-2}.$$

At the jth differentiation we get

$$P^{(j)}(x) = j!c_j + (j+1)!c_{j+1}x + \cdots + \cdots + K(K-1)\cdots(K-j+1)c_K x^{K-j}$$

for $j \le K$. In particular, letting $x = 0$ above yields

$$P^{(j)}(0) = j!c_j \quad \text{for } 1 \le j \le K.$$

Since at each differentiation the polynomial loses one degree, at the Kth differentiation $P^{(K)}$ has degree 0 (it is the constant $K!c_K$). Since the derivative of a constant is 0, we get

$$P^{(j)} = 0 \quad \text{for } j > K.$$

This completes the proof of Lemma 4.3.

We apply Lemma 4.3 to f_n to get:

Lemma 4.4 *We have that $f_n^{(j)}(0)$ and $f_n^{(j)}(\pi)$ are integers for all $j \ge 1$.*

We now prove Lemma 4.4. Applying Lemma 4.3 to the polynomial $n!f_n$, we get

$$n!f_n^{(j)}(0) = j!c_j$$

for $j = 1, \dots, 2n$ since the degree of f_n is $2n$. Since $c_j = 0$ for $j < n$, we have that

$$f_n^{(j)}(0) = 0 \quad \text{for } 1 \le j < n.$$

Note that if $j \ge n$, then $j!/n!$ is an integer, and so

$$f_n^{(j)}(0) = \frac{j!}{n!}c_j$$

are all integers for $j = n, \dots, 2n$. For j outside this range, $f_n^{(j)}(0)$ is 0. In summary, we have shown that $f_n^{(j)}(0)$ is an integer for every $j \ge 1$.

Moreover,

$$\begin{aligned} f_n(\pi - x) = f_n(a/b - x) &= \frac{(a/b - x)^n(a - b(a/b - x))^n}{n!} \\ &= \frac{(a/b - x)^n(bx)^n}{n!} = f_n(x). \end{aligned}$$

That is, f_n is symmetric with respect to π. By the chain rule we have, for all derivatives,

$$f_n^{(k)}(x) = (-1)^k f_n^{(k)}(\pi - x).$$

In particular, letting $x = 0$, we get

$$f_n^{(k)}(0) = (-1)^k f_n^{(k)}(\pi).$$

Therefore, $f_n^{(k)}(\pi)$ are also integers for any $k \ge 1$. This completes the proof of Lemma 4.4.

We define the function g_n by

$$g_n = f_n - f_n^{(2)} + f_n^{(4)} - \cdots + (-1)^n f_n^{(2n)}.$$

We have, by the product rule,

$$\begin{aligned} \big(g_n'(x)\sin x - g_n(x)\cos x\big)' &= g_n''(x)\sin x + g_n'(x)\cos x - g_n'(x)\cos x + g_n(x)\sin x \\ &= g_n''(x)\sin x + g_n(x)\sin x. \end{aligned}$$

Note that

$$\begin{aligned} g_n + g_n'' = \big(&f_n - f_n^{(2)} + f_n^{(4)} - \cdots + (-1)^n f^{(2n)}\big) \\ &+ \big(f_n^{(2)} - f_n^{(4)} + \cdots + (-1)^n f^{(2n)} + (-1)^n f_n^{(2n+2)}\big). \end{aligned}$$

There is cancellation of all terms except for the first and the last:

$$g_n + g_n'' = f_n + (-1)^n f_n^{(2n+2)}.$$

But f_n is a polynomial with degree $2n$, and therefore, $f_n^{(2n+2)} = 0$ and

$$g_n + g_n'' = f_n.$$

This yields

$$\left(g_n'(x)\sin x - g_n(x)\cos x\right)' = \left(g_n''(x) + g_n(x)\right)\sin x = f_n(x)\sin x.$$

By the fundamental theorem of Calculus,

$$\int_0^{\pi} f_n(x)\sin x\,dx = g_n'(x)\sin x - g_n(x)\cos x\Big]_0^{\pi} = g_n(\pi) + g_n(0).$$

Since $f_n^{(k)}(0)$ and $f_n^{(k)}(\pi)$ are integers for all k, $g_n(\pi)$ and $g_n(0)$ are also integers, and so is $\int_0^{\pi} f_n(x)\sin x\,dx$. It is easy to check that f_n and sin are positive on $[0,\pi]$. Moreover, $x^n \le \pi^n$ and $(a-bx)^n \le a^n$ for x in $[0,\pi]$. Thus,

$$f_n(x) < \frac{\pi^n a^n}{n!} \quad \text{for } x \in [0,\pi].$$

Therefore,

$$0 \le f_n(x)\sin x < \frac{\pi^n a^n}{n!} \quad \text{for } x \in [0,\pi].$$

We now integrate between 0 and π to get

$$0 \le \int_0^{\pi} \sin x f_n(x)\,dx \le \frac{\pi^{n+1}a^n}{n!} < 1$$

by our initial choice of n. Thus, $\int_0^{\pi} \sin x f_n(x)\,dx$ is a positive integer strictly less than 1. It must be 0. As we will now see, this is impossible and provides a contradiction. Since $0 \le f_n(x)\sin x$ for x in $[0,\pi]$, we have

$$\int_0^{\pi} f_n(x)\sin x\,dx \ge \int_{\pi/4}^{3\pi/4} f_n(x)\sin x\,dx.$$

The minimum of sin on $[\pi/4, 3\pi/4]$ is $\sqrt{2}/2$. It is easy to check that f_n increases on $[0, \frac{n}{n+1}\pi]$ and decreases on $[\frac{n}{n+1}\pi, \pi]$. Thus, the minimum of f_n on $[\pi/4, 3\pi/4]$ is at $\pi/4$ (since $\frac{n}{n+1}\pi > \pi/4$ for all naturals n). Therefore,

$$\int_0^{\pi} f_n(x)\sin x\,dx \ge \int_{\pi/4}^{3\pi/4} f_n(x)\sin x\,dx \ge f_n(\pi/4)\frac{\sqrt{2}}{2}\frac{\pi}{4} > 0,$$

where the last inequality comes from the fact that f_n is 0 only at 0 and π. Thus, we have our contradiction: $\int_0^{\pi} \sin x f_n(x)\,dx$ is 0 and is not 0. Our initial assumption that π is a rational cannot be true. This completes the proof that π is irrational.

Exercises

1. Show that for any real x, there is a natural n such that

$$\frac{x^n}{n!} < 1.$$

2. (a) Show that the maximum of f_n occurs at $\frac{n}{n+1}\pi$.
   (b) Show that f_n is 0 only for $x = 0$ and $x = \pi$.
3. Show that a rational number is algebraic but that the converse is not true.

4. Show that if x^2 is irrational, so is x. Is the converse true?
5. In this exercise we show that $\exp(r)$ is irrational for every rational $r \neq 0$. We first show that $\exp(k)$ is irrational for every natural k. Let

$$h_n(x) = \frac{x^n(1-x)^n}{n!}.$$

Let k be a fixed natural number and define

$$H_n(x) = k^{2n}h_n(x) - k^{2n-1}h_n'(x) + k^{2n-2}h_n^{(2)}(x) - \cdots + h^{(2n)}(x).$$

(a) Show that

$$\frac{d}{dx}\big(\exp(kx)H_n(x)\big) = k^{2n+1}\exp(kx)h_n(x).$$

(b) Integrate the equality above for x between 0 and 1 to get

$$\exp(k)H_n(1) - H_n(0) = k^{2n+1}\int_0^1 \exp(kx)h_n(x)\,dx.$$

(c) By contradiction, assume that $\exp(k)$ is a rational a/b where a and b are naturals. Show that

$$a\exp(k)H_n(1) - bH_n(0) = bk^{2n+1}\int_0^1 \exp(kx)h_n(x)\,dx.$$

(d) Show that there is N such that for all $n \geq N$ and all x in [0, 1], we have

$$bk^{2n+1}\exp(kx)h_n(x) = bk^{2n+1}\exp(kx)\frac{x^n(1-x)^n}{n!} < 1/2.$$

(e) Explain why $a\exp(k)H_n(1) - bH_n(0)$ is a positive integer.
(f) Use (d) and (e) to find a contradiction. Thus, $\exp(k)$ is not a rational.
(g) Show that $\exp(r)$ is not a rational if $r \neq 0$ is a rational.

6. Show that if $r \neq 1$ is a rational, then $\ln r$ is irrational. Use Exercise 5.

Chapter 5
Continuity, Limits, and Differentiation

5.1 Continuity

We start with a definition.

> **Continuous functions**
> Assume that the function f is defined on some set D. Let a be in D. f is said to be continuous at a if for any sequence a_n in D that converges to a, we have that $f(a_n)$ converges to $f(a)$.

Note that for ANY a_n converging to a, we MUST have $f(a_n)$ converging to $f(a)$ in order for f to be continuous at a.

Example 5.1 The function $f(x) = x^2$ is continuous everywhere on the reals. To prove that, take a a real number. Let a_n be any sequence of reals converging to a. Since the product of two converging sequences converges to the product of the limits, we have that $f(a_n) = a_n^2 = a_n a_n$ converges to a^2. But a^2 is also $f(a)$. Hence, f is continuous at a, and we are done.

In order to show that a function is not continuous at a, it is enough to find ONE sequence a_n in D with the following properties: a_n converges to a, but $f(a_n)$ does not converge to $f(a)$. We do such an example next.

Example 5.2 Consider the function g defined on $D = [-1, 1]$ by

$$g(x) = |x|/x \quad \text{for } x \neq 0 \quad \text{and} \quad g(0) = 1.$$

Hence, $g(x) = -1$ for x in $[-1, 0)$ and $g(x) = 1$ for x in $[0, 1]$. Intuitively, this function has a jump at 0 and so cannot be continuous at 0. See the graph below. We now prove this. Let $a_n = -1/n$. This sequence is in D and converges to 0. Since a_n is in $[-1, 0]$ for all $n \geq 1$, we have $g(a_n) = -1$. Hence, $g(a_n)$ converges to -1. But $g(0) = 1$. Therefore, g is not continuous at 0. See the graph of g in Fig. 5.1.

R.B. Schinazi, *From Calculus to Analysis*,
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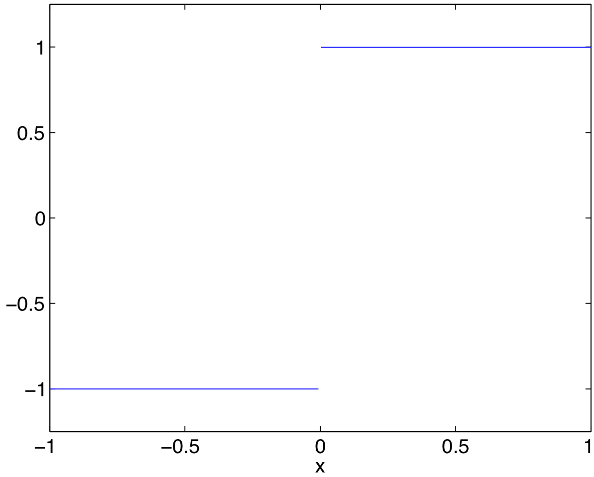

Fig. 5.1 This is the graph of $g(x) = |x|/x$

Example 5.3 The absolute value function is continuous on the reals.

Let a be a real, and a_n a sequence of reals converging to a. Recall that, as a consequence of the triangle inequality, we have for all reals x and y:

$$0 \leq \big||x| - |y|\big| \leq |x - y|.$$

Hence,

$$0 \leq \big||a_n| - |a|\big| \leq |a_n - a|.$$

By the squeezing principle, $||a_n| - |a||$ converges to 0. Hence, $|a_n| - |a|$ converges to 0, and therefore $|a_n|$ converges to $|a|$ (why?). Thus, the absolute value function is continuous at every a.

We will now define the usual operations on functions. Given that f and g are defined at a, we have that the function $f + g$ is defined as

$$(f + g)(a) = f(a) + g(a),$$

we define fg as

$$(fg)(a) = f(a)g(a),$$

and if $g(a) \neq 0$, we define f/g as

$$(f/g)(a) = f(a)/g(a).$$

The following properties allow us to construct many continuous functions from basic ones.

Operations on continuous functions
Let the functions f and g be defined on D and be continuous at a in D. Then:

$f + g$ is continuous at a;

fg is continuous at a;

if $g(a) \neq 0$, then f/g is continuous at a.

The proofs are direct consequences of the operations on limits that we recall below.

Operations on limits
Assume that the sequences a_n and b_n converge, respectively, to a and b.

(i)
$$\lim_{n\to\infty}(a_n + b_n) = a + b.$$

(ii)
$$\lim_{n\to\infty} a_n b_n = ab.$$

(iii) Assume that b_n is never 0 and that b is not 0. Then

$$\lim_{n\to\infty} a_n/b_n = a/b.$$

(iv) Assume that $a_n \leq b_n$ for all n. Then $a \leq b$.

We now prove that if f and g are continuous at a, so are $f + g$ and fg. Take any sequence a_n in I that converges to a. We know that

$$\lim_{n\to\infty} f(a_n) = f(a) \quad \text{and} \quad \lim_{n\to\infty} g(a_n) = g(a).$$

Hence, by the operations on limits,

$$\lim_{n\to\infty}\big(f(a_n) + g(a_n)\big) = f(a) + g(a) \quad \text{and} \quad \lim_{n\to\infty}\big(f(a_n)g(a_n)\big) = f(a)g(a).$$

For the ratio property, we first need to show that $g(a_n)$ is never 0 for n large enough since $g(a_n)$ converges to $g(a) \neq 0$. Assume that $g(a) < 0$. Then pick $\epsilon = -g(a)/2 > 0$. There is N such that if $n \geq N$, then

$$\big|g(a_n) - g(a)\big| < \epsilon = -g(a)/2.$$

Hence,

$$g(a_n) < g(a) - g(a)/2 = g(a)/2 < 0 \quad \text{for all } n \geq N.$$

That is, $g(a_n)$ remains negative for all $n \geq N$. Similarly, if we assume that $g(a) > 0$, we can show that there is a natural N_1 such that $g(a_n) > 0$ for all $n \geq N_1$. This allows us to define the sequence $f(a_n)/g(a_n)$ for $n \geq N$, and by (iii) we have

$$\lim_{n\to\infty} f(a_n)/g(a_n) = f(a)/g(a).$$

This completes the proof of the operations on continuous functions.

A *polynomial* is a function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where n is a nonnegative integer, and a_i, $0 \le i \le n$, are real numbers.

Example 5.4 A polynomial is continuous everywhere on the reals.

We first take care of power functions. Let k be a natural number, and let $f_k(x) = x^k$. Let P be the set of natural numbers k such that f_k is continuous everywhere. Let a be a real number, and a_n be a sequence converging to a. Then, of course, $f_1(a_n) = a_n$ converges to $f_1(a) = a$. Thus, 1 belongs to P. Assume that k belongs to P. Then $a_n^{k+1} = a_n^k a_n$ converges to $a^k a = a^{k+1}$ by (ii). By induction, we see that for every natural number k, the function f_k is continuous everywhere.

Consider now the function $g_k(x) = b_k x^k$. Then by (ii) we see that g_k is continuous for all naturals k and all reals b_k. Finally, a polynomial P is always a sum $P = g_0 + g_1 + g_2 + \cdots + g_k$. Using (i) and an induction proof, one can show that P is continuous everywhere. The details are left as an exercise.

Example 5.5 Consider the function h defined by $h(x) = 1/x$ for all $x \ne 0$. Let $D = (-\infty, 0) \cup (0, \infty)$; h is defined on D. We now show that h is continuous everywhere on D.

Let a be in D. Let a_n be in D and such that a_n converges to a. Then by (iii) we have that $1/a_n$ converges to $1/a$. That is, $h(a_n)$ converges to $h(a)$, and h is continuous at a.

Example 5.6 The function square root is continuous on $[0, \infty)$.

First take $a = 0$. Let x_n be in D and converging to 0. For any $\epsilon > 0$, there is N such that $|x_n| < \epsilon^2$ for $n \ge N$. Since x_n is positive, $|x_n| = x_n$, and since square root is an increasing function on D,

$$0 \le \sqrt{x_n} < \sqrt{\epsilon^2} = |\epsilon| = \epsilon.$$

This proves that the square root function is continuous at 0.

Take now $a > 0$. Assume that y_n is in D and converges to a. For any $\epsilon > 0$, there is N such that if $n \ge N$, we have

$$|y_n - a| < \epsilon\sqrt{a}.$$

Observe now that

$$\left|\sqrt{y_n} - \sqrt{a}\right| = \frac{|y_n - a|}{\sqrt{y_n} + \sqrt{a}}.$$

Since

$$\sqrt{y_n} + \sqrt{a} \ge \sqrt{a},$$

and the inverse function is decreasing on $(0, \infty)$, we have

$$\frac{1}{\sqrt{y_n} + \sqrt{a}} \le \frac{1}{\sqrt{a}}.$$

Therefore,

$$\left|\sqrt{y_n} - \sqrt{a}\right| = \frac{|y_n - a|}{\sqrt{y_n} + \sqrt{a}} \leq \frac{|y_n - a|}{\sqrt{a}}.$$

Hence, for $n \geq N$, we have

$$\left|\sqrt{y_n} - \sqrt{a}\right| \leq \frac{|y_n - a|}{\sqrt{a}} < \frac{\epsilon\sqrt{a}}{\sqrt{a}} = \epsilon.$$

This concludes the proof that the square root function is continuous at a.

Another important operation is the composition. Let f be defined at a, and g be defined at $f(a)$. We define the function $g \circ f$ at a by

$$g \circ f(a) = g\big(f(a)\big).$$

Continuity and composition of functions
Let f be defined on D, and let g be defined on the range C of f:

$$C = \left\{f(x) : x \in D\right\}.$$

Assume that f is continuous at $a \in D$ and that g is continuous at $f(a)$ in C. Then $g \circ f$ is continuous at a.

To prove the claim above, take a sequence a_n in D that converges to a. Since f is continuous at a, $f(a_n)$ converges to $f(a)$. Since g is continuous at $f(a)$ and $f(a_n)$ is a sequence in C that converges to $f(a)$, we have that $g(f(a_n))$ converges to $g(f(a))$. This shows that $g \circ f$ is continuous at a, and we are done.

Example 5.7 Let g be continuous on the reals. Let h be defined by $h(x) = g(x^2)$. Then h is continuous everywhere.

Let f be defined on the reals by $f(x) = x^2$. We have $h = g \circ f$. Since g and f are known to be continuous everywhere, so is h, and we are done.

Let $a < b$ be two reals. We denote the set of all reals x such that $a \leq x \leq b$ by $[a, b]$. The set $[a, b]$ is called a closed and bounded interval. It is said to be closed because the two bounds a and b are in the set. Intervals are the only sets of reals with no holes. This property is crucial for the next result. See Sect. 5.4 for a formal definition of intervals.

Intermediate Value Theorem
Assume that the function f is continuous at every point of the interval $[a, b]$. Let c be strictly between $f(a)$ and $f(b)$. There exists d in (a, b) such that $f(d) = c$.

We start by proving a particular case of the intermediate value theorem (IVT). Assume that $c = 0$ and that $f(a) < 0 < f(b)$. We need to show that the equation $f(x) = 0$ has at least one solution. The proof that we now write is interesting because not only it proves the existence of a solution, but also it provides an algorithm to find the solution up to an arbitrarily small error. We construct inductively two sequences a_n and b_n such that

$$f(a_n) \leq 0, \qquad f(b_n) \geq 0, \quad a_n \leq a_{n+1} \leq b_{n+1} \leq b_n, \text{ and } b_n - a_n \leq \frac{b_1 - a_1}{2^n}.$$

Let $a_1 = a$ and $b_1 = b$. Compute $f(\frac{a_1+b_1}{2})$.

If $f(\frac{a_1+b_1}{2}) < 0$, set $a_2 = \frac{a_1+b_1}{2}$ and $b_2 = b_1$.
If $f(\frac{a_1+b_1}{2}) > 0$, set $b_2 = \frac{a_1+b_1}{2}$ and $a_2 = a_1$.
If $f(\frac{a_1+b_1}{2}) = 0$, set $a_2 = b_2 = \frac{a_1+b_1}{2}$.

Note that in all cases we have $f(a_2) \leq 0$, $f(b_2) \geq 0$, $a_1 \leq a_2 \leq b_2 \leq b_1$, and $b_2 - a_2 \leq \frac{b_1 - a_1}{2}$.

Assume now that sequences a_k and b_k are defined up to $k = n$ and that

$$a_{n-1} \leq a_n \leq b_n \leq b_{n-1} \quad \text{and} \quad b_n - a_n \leq \frac{b_1 - a_1}{2^n}.$$

Then,

If $f(\frac{a_n+b_n}{2}) < 0$, set $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$.
If $f(\frac{a_n+b_n}{2}) > 0$, set $b_{n+1} = \frac{a_n+b_n}{2}$ and $a_{n+1} = a_n$.
If $f(\frac{a_n+b_n}{2}) = 0$, set $a_{n+1} = b_{n+1} = \frac{a_n+b_n}{2}$.

Since $a_n \leq b_n$, we have that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. Moreover, in all three cases we have

$$b_{n+1} - a_{n+1} \leq \frac{b_n - a_n}{2} \leq \frac{b_1 - a_1}{2^{n+1}}.$$

By induction we have constructed sequences a_n and b_n with the prescribed properties. Since

$$0 \leq b_n - a_n \leq \frac{b_1 - a_1}{2^n},$$

by the squeezing principle $b_n - a_n$ converges to 0 (why?). We also have that a_n is an increasing sequence bounded above by b_1 (why?), so it must converge. Similarly, b_n is a decreasing sequence bounded below by a_1, so it converges as well. Moreover, the two limits must be the same, and we denote the limit by d. Note that by construction

$$f(a_n) \leq 0 \quad \text{and} \quad f(b_n) \geq 0.$$

Letting n go to infinity in the two inequalities above and using that f is continuous at d (why?), we get

$$f(d) \leq 0 \quad \text{and} \quad f(d) \geq 0.$$

Thus, $f(d) = 0$, and we have proved the intermediate value theorem in the particular case $c = 0$.

We now turn to the general case. Let c be strictly between $f(a)$ and $f(b)$. Assume that $f(a) < c < f(b)$; the other case is treated similarly. Let the function g be defined by

$$g(x) = f(x) - c.$$

Note that g is also continuous on $[a, b]$. We have $g(a) < 0$ and $g(b) > 0$. According to the particular case of the IVT that we already proved, there is d in (a, b) such that $g(d) = 0$. This implies that $f(d) = c$, and the IVT is proved.

Example 5.8 We use the algorithm above to find an approximation of $\sqrt{2}$. Define $f(x) = x^2$. Note that $f(1) = 1$ and $f(2) = 4$ and that f is continuous on $[1, 2]$ (it is continuous everywhere of course, so also on $[1, 2]$). So by the IVT there is d in $(1, 2)$ such that $f(d) = 2$. That is, $d^2 = 2$ and $d = \sqrt{2}$. Let $a_1 = 1$ and $b_1 = 2$. We have $f(3/2) = 9/4 > 2$, so $a_2 = 1$ and $b_2 = 3/2$. The midpoint of 1 and 3/2 is 5/4, and $f(5/4) = 25/16 < 2$. Thus, $a_3 = 5/4$ and $b_3 = 3/2$. That is, $\sqrt{2}$ is in (a_3, b_3). An approximate value of $\sqrt{2}$ is then $(a_3 + b_3)/2 = 11/8$. The error is at most $(b_3 - a_3)/2 = (b_1 - a_1)/2^4 = 1/16$.

Example 5.9 Let g be defined by

$$g(x) = -1 \quad \text{for } x \in [-1, 0) \quad \text{and} \quad g(x) = 1 \quad \text{for } x \in (0, 1].$$

The IVT does not apply: observe, for instance, that 0 is between -1 and 1, but that there is no d such that $g(d) = 0$. The problem is that g is defined and continuous on $[-1, 0) \cup (0, 1]$ which is not an interval! It has a hole at 0. This is why the IVT cannot be applied.

We now turn to another important theorem concerning continuous functions.

Extreme Value Theorem
Assume that the function f is continuous at every point of the interval $[a, b]$. Then f attains both a minimum and a maximum value. That is, there are real numbers c and d in $[a, b]$ such that for all x in $[a, b]$,

$$f(c) \le f(x) \le f(d).$$

We prove this theorem in two steps. In the first step we prove that a continuous function on $[a, b]$ must be bounded. That is, we want to show that there is M such that for all x in $[a, b]$,

$$\left| f(x) \right| < M.$$

We do a proof by contradiction. Assume that for every natural n, there is x_n in $[a, b]$ such that

$$\left| f(x_n) \right| > n.$$

Since x_n is always in $[a, b]$, the Bolzano–Weierstrass theorem applies, and there is a subsequence x_{n_k} that converges to some ℓ. It is easy to see that ℓ is in $[a, b]$ (why?). Since f is continuous at ℓ, so is $|f|$ (why?). Hence, $|f(x_{n_k})|$ converges to $|f(\ell)|$. On the other hand,

$$\left|f(x_{n_k})\right| > n_k.$$

That is, $|f(x_{n_k})|$ is unbounded. This is a contradiction, a convergent sequence must be bounded. Therefore, f is bounded.

In our second step we show that f attains a maximum (that f attains a minimum is similar and left as an exercise). Consider

$$A = \left\{f(x) : x \in [a, b]\right\}.$$

That is, A is the range of f. Note that by the first step, A is bounded above by M. Clearly, A is not empty (why?). By the fundamental property of the reals, A has a least upper bound m. As shown in Sect. 2.1, there is a sequence y_n in A that converges to m. Since y_n is in A, there is z_n in $[a, b]$ such that $y_n = f(z_n)$. Hence, there is a sequence z_n in $[a, b]$ such that $f(z_n)$ converges to m. By Bolzano–Weierstrass (since z_n is bounded), there is a subsequence z_{n_k} that converges to some d in $[a, b]$. Since f is continuous at d, $f(z_{n_k})$ converges to $f(d)$. But $f(z_{n_k})$ converges to m as well since $f(z_{n_k})$ is a subsequence of $f(z_n)$. Hence, we must have $f(d) = m$. This proves that f attains its maximum at d.

Example 5.10 The extreme value theorem applies to continuous functions on closed and bounded intervals. That is, intervals of the type $[a, b]$. Consider $f(x) = 1/x$ for x on (0, 1]. The function f is continuous on (0, 1]. However, it is not bounded. For every n,

$$f\big(1/(n + 1)\big) = n + 1 > n.$$

Hence, f cannot attain its maximum. The EVT does not apply since (0, 1] is not closed at 0.

For most of our applications, we will use the definition of continuity using sequences. However, the following definition of continuity is also useful.

Another definition of continuity
Let f be defined on D. Then, f is continuous at $a \in D$ if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that if x is in D and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

We want to show that the statement A: 'f is continuous at a' is equivalent to the statement B: 'for every $\epsilon > 0$, there is a $\delta > 0$ such that if x is in D and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$'.

Assume non B. That is, there is $\epsilon > 0$ such that for all $\delta > 0$, there is x in D such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$. Since this must be true for all $\delta > 0$, we may pick $\delta = 1/n$. There is x in D such that

$$|x - a| < 1/n \quad \text{and} \quad \left|f(x) - f(a)\right| \geq \epsilon.$$

In fact x is found once $\delta = 1/n$ is fixed, therefore x depends on n, and we will denote it by x_n. Since we may do this for any $\delta = 1/n$, we get a sequence x_n in D with

$$|x_n - a| < 1/n \quad \text{for every } n \geq 1.$$

In particular, x_n converges to a. On the other hand,

$$\left|f(x_n) - f(a)\right| \geq \epsilon.$$

Hence, $f(x_n)$ does not converge to $f(a)$, and f is not continuous at a. This shows that non B implies non A. This is logically equivalent to A implies B.

Assume B. Fix $\epsilon > 0$. There is $\delta > 0$ such that if D is in I and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Let a_n be a sequence in D that converges to a. There is N such that if $n \geq N$, then

$$|a_n - a| < \delta.$$

By B this implies

$$\left|f(a_n) - f(a)\right| < \epsilon \quad \text{for } n \geq N.$$

That is, $f(a_n)$ converges to $f(a)$. The function f is continuous at a. We have shown that B implies A. This completes the proof that the two continuity definitions are equivalent.

Application 5.1 Assume that g is defined on D and continuous at $a \in D$. Suppose that $g(a) > 0$. Then, there exists $\delta > 0$ such that if x is in D and if $|x - a| < \delta$, then

$$g(x) > g(a)/2.$$

We apply the second continuity definition with $\epsilon = g(a)/2 > 0$. There is $\delta > 0$ such that if x is in D and $|x - a| < \delta$, then

$$\left|g(x) - g(a)\right| < \epsilon = g(a)/2.$$

In particular,

$$g(x) - g(a) > -g(a)/2.$$

That is,

$$g(x) > g(a)/2$$

for all x in D such that $|x - a| < \delta$.

Exercises

1. We complete the proof that a polynomial is continuous everywhere in Example 5.2. Show that for any natural n and any continuous functions $g_1, g_2, \ldots, g_n$, we have that $g_1 + g_2 + \cdots + g_n$ is continuous.
2. Show that $g(x) = \frac{x}{1+x^2}$ is continuous everywhere.
3. Prove that a rational function (that is, the ratio of two polynomials) is continuous where it is defined.
4. Show that the number d defined in the IVT is strictly between a and b.
5. Is the number d defined in the IVT unique? Prove it or give a counterexample.
6. (a) Show that the equation $x^3 + x + 5 = 0$ has at least one solution.
   (b) Find an approximation of a solution good to the first decimal.
   (c) Can you generalize the result in (a)?
7. Assume that f is continuous on $[-2, 2]$ and that $f(-2) > -2$, $f(2) < 2$. Show that the equation $f(x) = x$ has at least one solution.
8. Let f and g be continuous on [0,1]. Assume that $f(0) < g(0)$ and $f(1) > g(1)$. Show that the equation $f(x) = g(x)$ has at least one solution.
9. Consider the function $f(x) = \frac{1}{1+x^2}$ on $\mathbf{R}$.
   (a) Show that f attains its maximum.
   (b) Does f have a minimum?
   (c) Can the EVT be applied in this case?
10. Complete the proof of the EVT. That is, assume that f is continuous on $[a, b]$ and show that f attains its minimum. (Recall that we proved that f is bounded below and above.)
11. Let f be defined on the interval I and continuous at $a \in I$.
    (a) Show that $|f|$ is continuous at a.
    (b) Show that f^2 is continuous at a.
12. Assume that g is defined on D and continuous at $a \in D$. Suppose that $g(a) < 0$. Show that there exists $\delta > 0$ such that if x is in D and if $|x - a| < \delta$, then
$$g(x) < 3g(a)/4.$$
13. Assume that g is a continuous function on D. Let a be in D and such that $g(a) > 0$. Show that if a_n is a sequence in D that converges to a, then there is N such that if $n \geq N$, then $g(a_n) > 0$.
14. (a) Let f be a continuous function on $\mathbf{R}$. Assume that $f(r) = 0$ for every rational r. Show that $f(x) = 0$ for every real x. (Use the density of the rationals.)
    (b) Let f and g be continuous functions on $\mathbf{R}$. Assume that $f(r) = g(r)$ for every rational r. Show that $f(x) = g(x)$ for every real x.
15. Let f be defined on $\mathbf{R}$ and suppose that there is a real $M > 0$ such that
$$\big|f(x) - f(y)\big| \leq M|x - y|$$
for all x and y. Show that f is continuous on $\mathbf{R}$.
16. Let f be defined on $[0, \infty)$ by
$$f(x) = \min\big(x, x^2\big).$$
    (a) Sketch the graph of f.

(b) Show that for all x and y,

$$\min(x,y) = \frac{1}{2}\big((x+y) - |x-y|\big).$$

(c) Let f and g be continuous on D. Define h as $h(x) = \min(f(x), g(x))$. Show that h is continuous on D.

17. Let a be a real. Define the function f by $f(x) = |x-a|$. Prove that f is continuous on $\mathbf{R}$.
18. Let f be a continuous function on $[0,1]$. Assume that $f \geq 0$.
    (a) Show that either there is c such that $f(c) = 0$ or there is $a > 0$ such that $f(x) \geq a$ for all x in $[0,1]$ (in the latter case, f is said to be bounded away from 0).
    (b) Give an example of a function $g \geq 0$ continuous on $(0,1]$ for which (a) does not hold.
19. Define the function f as follows. If x is a rational, $f(x) = 1$. If x is not rational, $f(x) = 0$. Show that f is nowhere continuous. That is, for any a in $\mathbf{R}$, the function f is not continuous at a. (Use that the rational numbers are dense in the real numbers and that the irrational numbers are also dense in the real numbers.)

5.2 Limits of Functions and Derivatives

We start with a definition.

Limit of a function
Assume that f is defined on a set $D \subset \mathbf{R}$. The function f is said to have a limit ℓ at a if for every sequence x_n in D such that $x_n \neq a$ for all $n \geq 1$ and such that x_n converges to a, we have

$$\lim_{n\to\infty} f(x_n) = \ell.$$

The limit of the function f at a is denoted by

$$\lim_{x\to a} f(x) = \ell.$$

Note that in order for the definition above to make sense, the point a and the set D must be such that there are sequences in D with no term equal to a and converging to a. For instance, if $D = \{1,2\}$ and $a = 1$, then there is no such sequence (why not?). This is why one usually defines the notion of limit point of a set before defining limits of functions, see the Exercises. Since we will be interested only in sets D that are intervals or union of intervals and in points a that are either in D or at the boundary of D, we omit the notion of limit point for the time being.

Example 5.11 Let f be defined on $D = \mathbf{R}$ by $f(x) = x^2$. Let $a = 1$. Does f have a limit at 1?

Let x_n be a sequence in $\mathbf{R}$ that converges to 1 and such that $x_n \neq 1$ for all $n \geq 1$. Then x_n^2 converges to 1 (why?). Hence, $f(x_n)$ converges to 1. This proves that the limit of f at 1 exists and is 1.

Example 5.12 Let g be defined on $D = (-\infty, 0) \cup (0, \infty)$ by

$$g(x) = \frac{|x|}{x}.$$

Does g have a limit at 0?

Intuitively, the answer is clearly no. The function g is -1 for $x < 0$ and 1 for $x > 0$. Depending on which side we approach 0, we get one limit or the other. We now prove this. Let $x_n = 1/n$. Then x_n is always in D, converges to 0, and is never 0. Since $x_n > 0$, we have $g(x_n) = 1$ for all $n \geq 1$. Hence, $g(x_n)$ converges to 1.

Now, let $y_n = -1/n$. Then y_n is always in D, converges to 0, and is never 0. Since $y_n < 0$, we have $g(y_n) = -1$ for all $n \geq 1$. Hence, $g(y_n)$ converges to -1.

That is, we have two sequences converging to 0 but such that $g(x_n)$ and $g(y_n)$ converge to different limits. Hence, the function g has no limit at 0.

Note that we picked two of our favorite sequences to prove that g has no limit at $a = 0$. But if we wanted to prove that g has a limit, we would need to work with a generic sequence x_n converging to a as was done in Example 5.11.

Example 5.13 Let h be defined on $D = (0, \infty)$ by

$$h(x) = \frac{\sqrt{x}}{x}.$$

Does h have a limit at 0?

By computing $h(x)$ for a few x near 0 $(0.1, 0.01, \ldots)$ it is easy to see that h grows without bound near 0 and therefore has no limit. We now prove it. Let $x_n = 1/n$. Then x_n is always in D, converges to 0, and is never 0. We have $h(x_n) = \sqrt{n}$. This sequence is unbounded and therefore does not converge. The function h has no limit at 0.

Remark Examples 5.12 and 5.13 give useful methods to prove that a function has no limit. Assume that we want to show that function g has no limit at a. One method, used in Example 5.12, is to find two sequences a_n and b_n in the domain of g converging to a (but never equal to a) and such that $g(a_n)$ and $g(b_n)$ converge to different limits.

The other method, used in Example 5.13, is to find a sequence a_n in the domain of g, converging to a and such that $g(a_n)$ does not converge.

The following alternative definition of a limit is sometimes useful.

Alternative definition of limit
The function f defined on D has a limit ℓ at a if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that if x is in D and $0 < |x - a| < \delta$, then $|f(x) - \ell| < \epsilon$.

Observe that this ϵ–δ definition of a limit is almost the same as the corresponding definition of continuity at the send of Sect. 5.1. The only differences are that $|x - a| > 0$ (which is the same as $x \neq a$) and the use of ℓ instead of $f(a)$. The proof that this second definition is equivalent to the first one is almost identical to the proof that the two definitions of continuity are equivalent. We leave it as an exercise.

There is a close relation between the notion of limit and the notion of continuity.

Continuity and limit
Assume that the function f is defined on D. Let a be in D. The function f is continuous at a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

Assume first that f is continuous at a. Take a sequence x_n in D such that for all $n \geq 1$, $x_n \neq a$ and x_n converges to a. By the definition of continuity at a we have

$$\lim_{n \to \infty} f(x_n) = f(a).$$

This proves that f has a limit $f(a)$ as x approaches a. This proves the direct implication.

For the converse, assume that the limit of f at a exists and is equal to $f(a)$. Use the alternative definition of limit to get: for every $\epsilon > 0$, there is a $\delta > 0$ such that if x is in D and $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Observe now that $|f(x) - f(a)| < \epsilon$ holds for $x = a$ as well. Hence, it holds for all x in D such that $|x - a| < \delta$. According to the second definition of continuity in Sect. 5.1, this proves that f is continuous at a.

Example 5.14 Consider a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where n is natural, and $a_0, a_1, \ldots, a_n$ are reals. For any real b, we have that

$$\lim_{x \to b} p(x) = p(b).$$

Since a polynomial is continuous everywhere, we know that the limit of p at b is $p(b)$, and we are done.

Example 5.15 Consider the function

$$f(x) = \frac{x^2 - 4}{x + 2} \quad \text{for } x \neq -2$$

and

$$f(-2) = 1.$$

This function is defined on all of **R**. Does it have a limit as x approaches -2?

Let x_n converge to -2 and never be equal to -2. Hence,

$$f(x_n) = \frac{x_n^2 - 4}{x_n + 2} = x_n - 2.$$

Since x_n converges to -2, we have that $x_n - 2$ converges to -4. Thus, the limit of f as x approaches -2 exists and is equal to -4. Note that we could have set $f(-2)$ equal to anything we like and still get the same result. The limit of the function at a does not depend on the value of the function at a.

Is f continuous at -2? Here the answer is no. We have that

$$\lim_{x \to -2} f(x) = -4 \neq f(-2).$$

But if we had set $f(-2) = -4$, then the conclusion would have been: f is continuous at -2. Hence, whether a function is continuous at a depends on what the value of the function at a is.

We will now introduce the notion of derivative for which the type of set the function is defined on is quite important. This is why we introduce the following definition.

Open intervals
An open interval is a set of the type (a, b), $(-\infty, a)$, (a, ∞), or $\mathbf{R} = (-\infty, \infty)$.

We are now ready to define derivatives.

The derivative
Let f be a function defined on an open interval I. The function f is said to be differentiable at $a \in I$ if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. In this case, the limit above is denoted by $f'(a)$. The function f' is called the derivative of f.

For a fixed a, let ϕ be defined by

$$\phi(h) = \frac{f(a+h) - f(a)}{h}$$

for $h \neq 0$. Note that the differentiability of f is equivalent to the existence of the limit of ϕ at 0. Note that since I is an open interval, if a is in I, there exists a $\delta > 0$ such that $(a+\delta, a-\delta) \subset I$ (see the exercises). If h is in $(-\delta, \delta)$, then $a+h$ is in $(a-\delta, a+\delta) \subset I$. Hence, ϕ is defined on $D = (-\delta, 0) \cup (0, \delta)$. If we let $h_n = \frac{\delta}{2n}$, then h_n is always in D, is never 0, and converges to 0. Hence, h_n is in the set of sequences in D converging to 0. Therefore, this set is not empty, and the question of a limit of ϕ at 0 is meaningful.

The following result is quite useful to compute derivatives. It formalizes the observation that the limit of a function at a does not depend on what happens at a.

Lemma *Assume that f and g are defined on $D = (-\delta, 0) \cup (0, \delta)$ for some $\delta > 0$. Assume also that*

$$f(x) = g(x) \quad \textit{for } x \neq 0$$

and that

$$\lim_{x \to 0} g(x) = \ell.$$

Then ℓ is also the limit of f as x approaches 0.

In order to prove the lemma, let x_n be a sequence in D that converges to 0 and such that $x_n \neq 0$ for all n. Since $x_n \neq 0$, we have

$$f(x_n) = g(x_n) \quad \text{for every } n.$$

That is, the sequences $f(x_n)$ and $g(x_n)$ are identical. Since $g(x_n)$ converges to ℓ, so does $f(x_n)$, and this proves that ℓ is the limit of f as x approaches 0. The lemma is proved.

Example 5.16 A constant function is differentiable, and its derivative is 0.

For a real c, let $g(x) = c$ for all x in $\mathbf{R}$. Then, for any real a,

$$\frac{g(a+h) - g(a)}{h} = 0 \quad \text{for all } h \neq 0.$$

Note that the functions on the left-hand side and right-hand side are equal except for $h = 0$. The r.h.s. is a constant and so has a limit as h goes to 0. By the lemma, so does the r.h.s. We get $g'(a) = 0$ for all a. This shows that g is differentiable everywhere and that its derivative is identically 0.

Example 5.17 The function $f(x) = x^2$ is defined on the open interval $I = (-\infty, \infty)$. Note that

$$f(a+h) - f(a) = (a+h)^2 - a^2 = h(2a+h).$$

Hence,

$$\frac{f(a+h) - f(a)}{h} = 2a + h \quad \text{for all } h \neq 0.$$

Note that the left-hand side is not defined at 0, but the right-hand side is. The limit on the r.h.s. is $2a$. By the lemma the limit as h goes to 0 is the same for both sides. Therefore, f is differentiable everywhere, and f' is defined by

$$f'(x) = 2x.$$

We generalize the preceding example.

The derivative of x^n
For any natural n, the function $f(x) = x^n$ is differentiable everywhere on $\mathbf{R}$, and the derivative is

$$f'(x) = nx^{n-1}.$$

For $n = 1$, we have $f(x) = x$ and

$$\frac{f(a+h) - f(a)}{h} = \frac{h}{h} = 1 \quad \text{for all } h \neq 0.$$

Hence, the limit, as h approaches 0, is 1. That is, f is differentiable everywhere, and $f'(x) = 1$.

The case $n = 2$ was taken care of in Example 5.16. We now turn to $n \geq 3$. Recall that

$$a^n - b^n = (a-b)\big(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\big).$$

Therefore,

$$\begin{aligned} f(a+h) - f(a) &= (a+h)^n - a^n \\ &= h\big((a+h)^{n-1} + (a+h)^{n-2}a + \cdots + (a+h)a^{n-2} + a^{n-1}\big). \end{aligned}$$

We have

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= (a+h)^{n-1} + (a+h)^{n-2}a + \cdots \\ &\quad + (a+h)a^{n-2} + a^{n-1} \quad \text{for all } h \neq 0. \end{aligned}$$

The right-hand side is a polynomial (and therefore a continuous function) in h. The limit, as h approaches 0, is the value of the polynomial for $h = 0$. Moreover, this polynomial is a sum of n terms

$$(a+h)^k a^{n-1-k} \quad \text{for } k = 0, 1, \ldots, n-1,$$

whose value for $h = 0$ is

$$a^k a^{n-1-k} = a^{n-1}.$$

Thus,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = na^{n-1}.$$

This proves the formula.

We now turn to a geometric interpretation of the derivative.

Geometric interpretation
Assume that f is defined on an open interval I and is differentiable at a in I. Then, near a the function f may be approximated by the linear function g,

$$g(x) = f(a) + f'(a)(x - a).$$

More precisely, we have that

$$\lim_{h \to 0} \frac{f(a+h) - g(a+h)}{h} = 0.$$

Moreover, the line whose equation is

$$y = g(x)$$

is called the tangent line to the graph of f at the point $(a, f(a))$.

We prove this by using the definition of g,

$$\frac{f(a+h) - g(a+h)}{h} = \frac{f(a+h) - f(a) - f'(a)h}{h} = \frac{f(a+h) - f(a)}{h} - f'(a),$$

whose limit is 0 as h goes to 0.

In the exercises it will be shown that the result above has a converse.

Example 5.18 The function $f(x) = |x|$ is not differentiable at 0.

We proved in Example 5.12 that $|h|/h$ has no limit at 0. Hence,

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \quad \text{does not exist.}$$

Therefore, f is not differentiable at 0. Graphically, this is so because the graph of the function has a corner at $(0, 0)$.

Note that the absolute value function is differentiable everywhere except at 0. See the exercises.

Example 5.19 The function $f(x) = \sqrt{x}$ is not differentiable at 0.

In Example 5.13, we proved that $\frac{\sqrt{h}}{h}$ does not converge as h goes to 0. Therefore,

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \quad \text{does not exist.}$$

That is, f is not differentiable at 0.

This can be seen on the graph by observing that the tangent at 0 is vertical. Note that the function square root is differentiable on all strictly positive reals. See the exercises.

We now turn to the relation between continuity and differentiability.

Differentiability implies continuity
Let f be defined on an open interval I and assume that f is differentiable at $a \in I$. Then f is continuous at a.

To prove this property, we use the alternative definition of limit. We set $\epsilon = 1$. Since f is differentiable at a, there is $\delta > 0$ such that if $0 < |h| < \delta$, then

$$\left| \frac{f(a+h) - f(a)}{h} - f'(a) \right| < 1.$$

We have

$$\begin{aligned} \left| \frac{f(a+h) - f(a)}{h} \right| &= \left| \frac{f(a+h) - f(a)}{h} - f'(a) + f'(a) \right| \\ &\leq \left| \frac{f(a+h) - f(a)}{h} - f'(a) \right| + \left| f'(a) \right|, \end{aligned}$$

where we are using the triangle inequality. Hence, for $0 < |h| < \delta$,

$$\left| \frac{f(a+h) - f(a)}{h} \right| < 1 + \left| f'(a) \right|,$$

and therefore,

$$\left| f(a+h) - f(a) \right| < |h|\big(1 + \left| f'(a) \right|\big).$$

Note that for $h = 0$, both sides of the inequality are 0. Hence, for $|h| < \delta$ (we are now including the value $h = 0$), we have

$$\left| f(a+h) - f(a) \right| \leq |h|\big(1 + \left| f'(a) \right|\big).$$

It is now easy to prove continuity. Let a_n be a sequence converging to a. Then $h_n = a_n - a$ converges to 0, and there is a natural N such that $|h_n| < \delta$ for $n \geq N$ (why?). Hence,

$$0 \leq \left| f(a_n) - f(a) \right| = \left| f(a+h_n) - f(a) \right| \leq |h_n|\big(1 + \left| f'(a) \right|\big).$$

By the squeezing principle $|f(a_n) - f(a)|$ converges to 0, and therefore $f(a_n)$ converges to $f(a)$. The proof that f is continuous at a is complete.

Note that Example 5.18 gives an example of a function (absolute value) which is continuous at 0 but not differentiable at 0. This shows that the converse of the property above does not hold.

In the next example we show how the notion of differentiability may be used to compute useful limits.

Example 5.20 Find

$$\lim_{x \to 0} \frac{\exp(x) - 1}{x}.$$

The function exp is differentiable, and its derivative is itself. So

$$(\exp)'(0) = \exp(0) = 1.$$

By the definition of differentiability,

$$\lim_{x\to 0} \frac{\exp(0+x) - \exp(0)}{x} = (\exp)'(0) = 1.$$

Since

$$\frac{\exp(0+x) - \exp(0)}{x} = \frac{\exp(x) - 1}{x},$$

we have

$$\lim_{x\to 0} \frac{\exp(x) - 1}{x} = 1.$$

Exercises

1. (a) Find a sequence in $D = (-\infty, 2) \cup (2, +\infty)$ that converges to 2.
   (b) Give an example of a set D and a point a such that there is no sequence x_n in D converging to a.
2. Let

$$f(x) = \frac{x^3+1}{x+1} \quad \text{for } x \neq -1,$$
$$f(-1) = c.$$

   (a) Does f have a limit at -1?
   (b) Is it possible to pick c so that f is continuous at -1?
3. Assume that f is defined on an open interval I that contains a. Let g be such that $g(x) = mx + b$ for all x and such that $g(a) = f(a)$. Moreover, assume that

$$\lim_{h\to 0} \frac{f(a+h) - g(a+h)}{h} = 0.$$

   (a) Prove that f is differentiable at a.
   (b) Show that $g(x) = f(a) + (x-a)f'(a)$.
4. (a) Assume that a is in (c,d). Find $\delta > 0$ such that $(a-\delta, a+\delta) \subset (c,d)$.
   (b) Is the property in (a) true if we have $[c,d)$ instead of (c,d)?
5. Show that if f is differentiable at a, then

$$\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{h} = 2f'(a).$$

6. Assume that $f(0) = 0$ and that f is differentiable at 0.
   (a) Find the limit of $f(x)/x$ as x goes to 0.
   (b) Find the limit of $f(x^2)/x$ as x goes to 0.
   (c) Give an example of f for which $f(x)/x^2$ has no limit as x goes to 0 and one for which it has a limit.

7. Assume that f is defined everywhere and that there is a constant C such that
$$\left|f(x)\right| \le Cx^2.$$
Prove that f is differentiable at 0.
8. Let f and g be defined on an open interval containing a. Assume that there is $d > 0$ such that $f(x) = g(x)$ for all x in $(a-d, a+d)$. Assume that g is differentiable at a. Show that f is also differentiable at a and that $f'(a) = g'(a)$.
9. Let $f(x) = |x|$.
   (a) Let $a < 0$. Show that there is $d > 0$ such that $f(x) = g(x)$ for all x in $(a-d, a+d)$ where $g(x) = -x$.
   (b) Show that f is differentiable at a. (Use Exercise 8.)
   (c) Let $b > 0$. Show that f is differentiable at b.
10. Assume that $a > 0$.
   (a) Show that if h_n converges to 0, then there exists a natural N such that $a + h_n > 0$ for $n \ge N$.
   (b) Show that
$$\sqrt{a+h_n} - \sqrt{a} = \frac{h_n}{\sqrt{a+h_n} + \sqrt{a}}.$$
   (c) Use (b) to show that the square root function is differentiable at $a > 0$.
11. Let $f(x) = \sin(1/x)$ for $x \neq 0$.
   (a) Find the limit of
$$f\left(\frac{1}{2n\pi}\right)$$
   as n goes to infinity.
   (b) Find the limit of
$$f\left(\frac{1}{2n\pi + \pi/2}\right)$$
   as n goes to infinity.
   (c) Use (a) and (b) to conclude that f has no limit at 0.
12. Let $g(x) = x\sin(1/x)$ for $x \neq 0$ and $g(0) = 0$.
   (a) Show that
$$0 \le \left|g(x)\right| \le |x|.$$
   (b) Show that g is continuous at 0.
   (c) Is g differentiable at 0? (You may use Exercise 10(c).)
13. For a natural n, let $k(x) = x^n \sin(1/x)$ for $x \neq 0$ and $k(0) = 0$.
   (a) Show that
$$0 \le \left|\frac{k(0+h) - k(0)}{h}\right| \le h^{n-1}.$$
   (b) Show that if $n \ge 1$, k is continuous at 0.
   (c) Show that if $n \ge 2$, k is differentiable at 0.

14. Use the method in Example 10 to show that
    (a)
    $$\lim_{x\to 0}\frac{\sin x}{x}=1.$$
    (b) What is
    $$\lim_{x\to 0}\frac{\cos x-1}{x}?$$
15. In Sect. 5.1 we have shown that f is continuous at $a\in I$ if and only if for every $\epsilon>0$, there is a $\delta>0$ such that if x is in I and $|x-a|<\delta$, then $|f(x)-f(a)|<\epsilon$.
    Imitate the proof in Sect. 5.1 to show that f has a limit ℓ at a if and only if for every $\epsilon>0$, there is a $\delta>0$ such that if x is in I and $0<|x-a|<\delta$, then $|f(x)-\ell|<\epsilon$.
16. The real number a is said to be a limit point of a set $D\subset\mathbf{R}$ if there is a sequence x_n *in* D such that $x_n\neq a$ for all $n\geq 1$ and that converges to a. Let
    $$D=\{x\in\mathbf{R}:0<x<1\}.$$
    (a) Is 0 a limit point of D?
    (b) Is 2 a limit point of D?
17. (a) Let $D=\{1,2\}$. Show that D has no limit points (see Exercise 16 for the definition).
    (b) Find and prove a statement that generalizes (a).
18. Let $\delta>0$. Then prove that 0 is a limit point of
    $$I=(-\delta,\delta).$$

5.3 Algebra of Derivatives and Mean Value Theorems

In this section we will prove several formulas that are useful to compute derivatives. Here are the first formulas.

Operations on differentiable functions
Assume that f and g are defined on an open interval I and are differentiable at $a\in I$. Let $c\in\mathbf{R}$ be a constant. Then

(i) cf is differentiable at a, and $(cf)'(a)=cf'(a)$.
(ii) $f+g$ is differentiable at a, and $(f+g)'(a)=f'(a)+g'(a)$.
(iii) fg is differentiable at a, and $(fg)'(a)=f'(a)g(a)+f(a)g'(a)$.
(iv) If $g(a)\neq 0$, then f/g is differentiable at a, and
$$(f/g)'(a)=\frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}.$$

We are going to use the operations on limits to prove these formulas. Assume that h_n converges to 0 and is never 0. We have

$$\frac{(cf)(a+h_n)-(cf)(a)}{h_n}=c\frac{f(a+h_n)-f(a)}{h_n}.$$

Since $\frac{f(a+h_n)-f(a)}{h_n}$ converges to $f'(a)$ and the product of a convergent sequence and a constant converges to the product of the constant and the limit, we get that

$$\frac{(cf)(a+h_n)-(cf)(a)}{h_n}$$

converges to $cf'(a)$. This proves that cf is differentiable at a and formula (i).

We turn to (ii):

$$\frac{(f+g)(a+h_n)-(f+g)(a)}{h_n}=\frac{f(a+h_n)-f(a)}{h_n}+\frac{g(a+h_n)-g(a)}{h_n}.$$

Since $\frac{f(a+h_n)-f(a)}{h_n}$ converges to $f'(a)$, $\frac{g(a+h_n)-g(a)}{h_n}$ converges to $g'(a)$, and the sum of two convergent sequences converges to the sum of the limits, we have that

$$\frac{f(a+h_n)-f(a)}{h_n}+\frac{g(a+h_n)-g(a)}{h_n}$$

converges to $f'(a)+g'(a)$. This proves that $f+g$ is differentiable at a and formula (ii).

To show (iii), we start with

$$\begin{aligned}&(fg)(a+h_n)-(fg)(a)\\&\quad=f(a+h_n)g(a+h_n)-f(a)g(a)\\&\quad=\big(f(a+h_n)-f(a)\big)g(a+h_n)+f(a)\big(g(a+h_n)-g(a)\big).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{(fg)(a+h_n)-(fg)(a)}{h_n}=&\frac{f(a+h_n)-f(a)}{h_n}g(a+h_n)\\&+f(a)\frac{g(a+h_n)-g(a)}{h_n}.\end{aligned}$$

Note that g must be continuous at a (since it is differentiable there). Thus, $g(a+h_n)$ converges to $g(a)$. Moreover, $\frac{f(a+h_n)-f(a)}{h_n}$ converges to $f'(a)$, $\frac{g(a+h_n)-g(a)}{h_n}$ converges to $g'(a)$. Using these facts together with the known operations on limits, we get that

$$\frac{(fg)(a+h_n)-(fg)(a)}{h_n}$$

converges to $f'(a)g(a)+f(a)g'(a)$. This proves (iii).

In order to prove (iv), observe first that since $g(a)\neq 0$ and g is continuous at a, there must be an interval around a where g is never 0. See Application 5.1

in Sect. 5.1. Hence, f/g is well defined near a. We have

$$\begin{aligned}\frac{f}{g}(a+h_n)-\frac{f}{g}(a) &= \frac{f(a+h_n)g(a)-f(a)g(a+h_n)}{g(a+h_n)g(a)}\\ &= \frac{(f(a+h_n)-f(a))g(a)-f(a)(g(a+h_n)-g(a))}{g(a+h_n)g(a)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{1}{h_n}\left(\frac{f}{g}(a+h_n)-\frac{f}{g}(a)\right)\\ &\quad=\frac{1}{g(a+h_n)g(a)}\left(\frac{f(a+h_n)-f(a)}{h_n}g(a)-f(a)\frac{g(a+h_n)-g(a)}{h_n}\right).\end{aligned}$$

Letting n go to infinity, we get

$$\lim_{n\to\infty}\frac{1}{h_n}\left(\frac{f}{g}(a+h_n)-\frac{f}{g}(a)\right)=\frac{1}{g(a)^2}\big(f'(a)g(a)-f(a)g'(a)\big).$$

This shows that f/g is differentiable at a and the formula in (iv).

Example 5.21 Polynomials are differentiable everywhere.

We know that f_k defined by $f_k(x)=x^k$ is differentiable everywhere. If c_k is a constant, by (i) $c_k f_k$ is differentiable everywhere. By repeated use of (ii) (see the exercises),

$$P=\sum_{k=0}^{n} c_k f_k$$

is differentiable everywhere. This proves that polynomials are differentiable everywhere.

Example 5.22 Rational functions are differentiable where they are defined.

A rational function $f=P/Q$ where P and Q are polynomials. The function f is defined where Q is not 0. Assume that $Q(a)\neq 0$. By Example 5.21, P is differentiable at a, and Q is differentiable at a. By (iv) P/Q is differentiable at a.

The range of a function f defined on I is the set $\{f(x) : x\in I\}$. Recall that if f is defined at a and g is defined at $f(a)$, then the function $g\circ f$ is defined at a by

$$(g\circ f)(a)=g\big(f(a)\big).$$

Chain rule

Assume that f is defined on some open interval I and differentiable at a. Assume that g is defined on some open interval J that contains the range of f and that g is differentiable at $f(a)$. Then $g\circ f$ is differentiable at a, and

$$(g\circ f)'(a)=f'(a)g'\big(f(a)\big).$$

To prove the chain rule, we approach f and g linearly near a. Define the function v for $h \neq 0$ by

$$v(h) = \frac{f(a+h) - f(a)}{h} - f'(a)$$

and $v(0) = 0$. Recall that since f is defined on an open interval, v is defined on an interval $(-\delta, \delta)$ for some $\delta > 0$ (see the observation following the definition of the derivative in the preceding section). Since f is differentiable at a, the limit of v at 0 is $0 = v(0)$. Hence, v is continuous at 0. Moreover,

$$f(a+h) = f(a) + hf'(a) + hv(h).$$

Similarly, we define the function w as

$$w(h) = \frac{g(f(a+h)) - g(f(a))}{h} - g'\big(f(a)\big) \quad \text{for } h \neq 0$$

and $w(0) = 0$. Again, w is well defined near 0. Since g is differentiable at $f(a)$, the limit of w at 0 is 0. Thus, w is continuous at 0. We have

$$g\big(f(a+h)\big) = g\big(f(a)\big) + hg'\big(f(a)\big) + hw(h).$$

Let h_n be a sequence converging to 0 and never equal to 0. We have

$$g\big(f(a+h_n)\big) = g\big(f(a) + h_n f'(a) + h_n v(h_n)\big).$$

Let

$$k_n = h_n f'(a) + h_n v(h_n).$$

Hence,

$$g\big(f(a+h_n)\big) = g\big(f(a) + k_n\big) = g\big(f(a)\big) + k_n g'\big(f(a)\big) + k_n w(k_n).$$

Thus,

$$\frac{g(f(a+h_n)) - g(f(a))}{h_n} = \frac{k_n}{h_n} g'\big(f(a)\big) + \frac{k_n}{h_n} w(k_n).$$

Note that

$$\frac{k_n}{h_n} = f'(a) + v(h_n).$$

Since h_n converges to 0, $v(h_n)$ converges to 0, and k_n converges to 0 as well. Thus, $\frac{k_n}{h_n}$ converges to $f'(a)$. Since w is continuous at 0 and k_n converges to 0, $w(k_n)$ converges to $w(0) = 0$, and $\frac{k_n}{h_n} w(k_n)$ converges to 0. Hence,

$$\lim_{n \to \infty} \frac{g(f(a+h_n)) - g(f(a))}{h_n} = f'(a) g'\big(f(a)\big).$$

This proves that $g \circ f$ is differentiable at a and the chain rule.

Example 5.23 Given a real $\alpha \neq 0$, consider the function f defined on $D = (0, \infty)$ by $f(x) = x^\alpha$. Show that f is differentiable everywhere on D and that

$$f'(b) = \alpha b^{\alpha - 1} \quad \text{for all } b > 0.$$

In Sect. 3.4 we defined

$$x^\alpha = \exp(\alpha \ln x).$$

We also stated that exp is differentiable on **R** and its derivative is itself. In Chap. 7 we will show that power series are differentiable on their open interval of convergence and that they can be differentiated term by term. Given this result, it is not difficult to show that the inverse function of exp, ln, is differentiable on $(0, \infty)$ and its derivative is $1/x$. This will be proved in the next section. Now we use these facts.

First observe that if g is defined by $g(x) = \alpha \ln x$, then

$$f = \exp \circ g.$$

Since exp is differentiable everywhere and g is differentiable on D, the chain rule may be applied at any $b > 0$ to show that f is differentiable on D and that

$$f'(b) = g'(b)\exp\big(g(b)\big) = \frac{\alpha}{b}\exp(\alpha \ln b) = \alpha \frac{b^\alpha}{b} = \alpha b^{\alpha-1}.$$

Example 5.24 Given a real $\alpha > 0$, consider the function f defined on **R** by $f(x) = \alpha^x$. Show that f is differentiable everywhere and that

$$f'(b) = \alpha^b \ln \alpha \quad \text{for all } b.$$

We use the same formula as in the preceding example:

$$\alpha^x = \exp(x \ln \alpha).$$

Let g be defined by $g(x) = x \ln \alpha$. Since exp and g are differentiable everywhere and $g'(x) = \ln \alpha$, by the chain rule we have

$$f'(b) = \ln \alpha \exp\big(g(b)\big) = \ln \alpha \exp(b \ln \alpha) = \alpha^b \ln \alpha.$$

We now turn to important applications of the notion of derivative.

Local extrema
Let f be defined on an open interval I. The function f is said to have a local minimum at a in I if there is a $\delta > 0$ such that $f(a) \le f(x)$ for every x in $(a-\delta, a+\delta)$. Likewise, f is said to have a local maximum at a if there is a $\delta > 0$ such that $f(a) \ge f(x)$ for every x in $(a-\delta, a+\delta)$.

The following is a useful result.

Local extrema and differentiability
Let f be defined on an open interval I. Assume that it has a local extremum at a and that f is differentiable at a. Then $f'(a) = 0$.

The proof is simple. Since I is open, there is a $\delta > 0$ such that $(a-\delta, a+\delta) \subset I$. Let $h_n = \frac{\delta}{2n}$. Then $a + h_n$ belongs to I for every $n \geq 1$. We assume that f has a local maximum at a (the local minimum case is similar). There is a $\delta_1 > 0$ such that if x belongs to $(a - \delta_1, a + \delta_1)$, then $f(x) \geq f(a)$. There is N such that if $n \geq N$, then $h_n < \delta_1$ (why?). Hence,

$$f(a + h_n) \leq f(a)$$

for $n \geq N$. This yields

$$\frac{f(a + h_n) - f(a)}{h_n} \leq 0.$$

By letting n go to infinity and using that f is differentiable at a, we get

$$f'(a) \leq 0.$$

We are now going to show that $f'(a) \geq 0$, and this will conclude the proof. Let $k_n = -h_n$. We have that $|k_n| = h_n < \delta_1$ for $n \geq N$. Hence,

$$f(a + k_n) \leq f(a)$$

for $n \geq N$. Using that $k_n < 0$, we get

$$\frac{f(a + k_n) - f(a)}{k_n} \geq 0.$$

Letting n go to infinity, we get $f'(a) \geq 0$. This concludes the proof.

We use the result above to prove the following generalized mean value theorem.

Cauchy Mean Value Theorem
Let f and g be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a real c in (a, b) such that

$$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c).$$

We define h on $[a, b]$ by

$$h(x) = \big(f(b) - f(a)\big)g(x) - \big(g(b) - g(a)\big)f(x).$$

Note that h is continuous on $[a, b]$ and differentiable on (a, b) (why?). We have

$$h'(x) = \big(f(b) - f(a)\big)g'(x) - \big(g(b) - g(a)\big)f'(x).$$

To prove the theorem, we only need to find c in (a, b) such that $h'(c) = 0$.

There are two cases. If h is constant on $[a, b]$, then we know that $h' = 0$ on (a, b). Hence, c may be any real in (a, b).

We now turn to the case where h is not constant. Note that

$$h(a) = h(b) = f(b)g(a) - f(a)g(b).$$

If h is not constant on $[a, b]$, there is at least one x_0 in (a, b) such that $h(x_0) \neq h(a)$. Assume that $h(x_0) < h(a)$ (the other case is similar). Since h is continuous on $[a, b]$, which is closed and bounded, the extreme value theorem applies, and there is c in $[a, b]$ such that for all x in $[a, b]$, we have

$$h(x) \geq h(c).$$

Hence, $h(c) \leq h(x_0) < h(a) = h(b)$. Therefore, c cannot be a or b, and it must be in the open interval (a, b). Since a minimum occurs at c for h and h is differentiable on the open interval (a, b), we must have $h'(c) = 0$. This completes the proof of the Cauchy mean value theorem.

An easy consequence of the preceding theorem is the following:

Lagrange Mean Value Theorem
Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a real c in (a, b) such that

$$f(b) - f(a) = (b - a)f'(c).$$

We apply the Cauchy mean value theorem with $g(x) = x$, which is continuous on $[a, b]$ and differentiable on (a, b). Hence, there is c in (a, b) such that

$$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c).$$

Since $g(x) = x$ and $g'(x) = 1$, we get

$$f(b) - f(a) = (b - a)f'(c).$$

This completes the proof of the Lagrange mean value theorem (LMVT).

Geometrically, the LMVT has a nice interpretation: there is at least one c in (a, b) where the line tangent to the graph of f (with slope $f'(c)$) has the same slope as the line passing through $(a, f(a))$ and $(b, f(b))$. Draw a picture!

We will apply the LMVT to obtain a number of important results from Calculus.

Rolle's Theorem
Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Assume also that $f(a) = f(b)$. Then there is a real c in (a, b) such that

$$f'(c) = 0.$$

This is an immediate application of the LMVT. There is c such that

$$f(b) - f(a) = (b - a)f'(c).$$

Since $f(a) = f(b)$, we have $f'(c) = 0$.

The following is a typical application of Rolle's theorem.

Example 5.25 Let f be a function twice differentiable everywhere and such that f'' is never 0. Then the equation $f(x)=0$ has at most two solutions.

By contradiction, assume that there are three solutions $a<b<c$. Since $f(a)=f(b)$, by Rolle's theorem there is a c_1 in (a,b) such that $f'(c_1)=0$. There is also c_2 in (b,c) such that $f'(c_2)=0$. Note that $c_1<c_2$. Since f' is differentiable, we may apply Rolle's theorem to f'. Using that $f'(c_1)=f'(c_2)=0$, there is c in (c_1,c_2) such that $f''(c)=0$. But f'' is never 0. We have a contradiction. The equation has at most two solutions.

Null derivative
Let f be differentiable on the open interval I. Then the function f is a constant on I if and only if $f'(x)=0$ for all x in I.

We already know that if f is a constant on I, then $f'=0$ on I. We prove the converse. Assume that $f'=0$ on I. Let $a<b$ in I. Then f is continuous on $[a,b]$ (why?) and differentiable on (a,b). The LMVT applies, and we get

$$f(b)-f(a)=f'(c)(b-a)$$

for some c in (a,b). Since $f'(c)=0$, we have $f(b)=f(a)$. Because this is true for any a and b, f is constant on I. We now turn to an important definition.

Increasing functions
Let f be defined on D. The function f is said to be increasing on D if for all $x<y$ in D, we have

$$f(x)\leq f(y).$$

It is said to be strictly increasing on D if for all $x<y$ in D, we have

$$f(x)<f(y).$$

We have a symmetric definition for decreasing functions.

Decreasing functions
The function f is said to be decreasing on D if for all $x<y$ in D, we have

$$f(x)\geq f(y).$$

It is said to be strictly decreasing on D if for all $x<y$ in D, we have

$$f(x)>f(y).$$

A function which is increasing or decreasing on D is said to be monotone on D.

Monotone differentiable functions
Let f be differentiable on the open interval I.

(a) If $f' > 0$ on I, then f is strictly increasing on I.
(b) If $f' < 0$ on I, then f is strictly decreasing on I.

We prove (a), and we leave (b) to the reader. Take $x < y$ in I. The LMVT may be applied to f on $[x, y]$ (why?). Hence, there is c in (x, y) such that

$$f(x) - f(y) = (x - y)f'(c).$$

Since $f' > 0$, we see that $(x - y)f'(c) < 0$, and so $f(x) - f(y) < 0$. Since this is true for all $x < y$ in I, the proof of (a) is complete.

Remark If $f' \geq 0$ on an open interval, the function f may be increasing or strictly increasing. See the exercises.

Example 5.26 The function $g(x) = 1/x$ is differentiable on $D = (-\infty, 0) \cup (0, \infty)$. The derivative of g is $g'(x) = -1/x^2 < 0$ for all x in D. However, g is not decreasing on D. For instance, $g(-1) < g(1)$. This does not contradict the preceding result: D is not an interval. The preceding result does show that g is strictly decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

Example 5.27 Let f be differentiable on the open interval I. Let a be in I, and $\delta > 0$ be such that $(a - \delta, a + \delta) \subset I$. Assume that if $a < x < a + \delta$, then $f'(x) < 0$ and if $a - \delta < x < a$, then $f'(x) > 0$. Show that f has a local maximum at a.

The proof is a simple application of the LMVT. Take x in $(a - \delta, a)$. Then there is c in (x, a) such that

$$f(a) - f(x) = f'(c)(a - x) > 0.$$

Hence, $f(a) > f(x)$ for x in $(a - \delta, a)$. Take y in $(a, a + \delta)$. There is d in (a, y) such that

$$f(a) - f(y) = f'(d)(a - y) > 0.$$

Hence, $f(a) > f(x)$ for all $x \neq a$ in $(a - \delta, a + \delta)$. A local maximum occurs at a.

Exercises

1. Let n be a natural. Let $f_1, f_2, \ldots, f_n$ be defined on an open interval I and differentiable at a in I. Show that $g = f_1 + f_2 + \cdots + f_n$ is differentiable at a.
2. Assume that f is defined on an open interval I, is differentiable at a in I, and $f(a) \neq 0$. Show that $1/f$ is differentiable at a and find a formula for $(1/f)'(a)$.
3. Assume that f and f' are differentiable on $\mathbf{R}$ and that for every x in $\mathbf{R}$, $f(x) + f''(x) = 0$. Show that $g = f^2 + (f')^2$ is a constant.

4. Assume that f and g are differentiable on an open interval I. Assume also that $f' = g'$ on I. Show that there is a constant c such that for all x in I,
$$f(x) = g(x) + c.$$
5. Let f be differentiable on an open interval I. Show that if $f' \geq 0$ on I, then f is increasing on I.
6. Give an example of a differentiable function which is strictly increasing on **R** but whose derivative is not strictly positive on **R**.
7. (a) Assume that $f'(a) = 0$. Does this necessarily imply that a local extremum occurs at a?
   (b) Draw the graph of a continuous function that has a local maximum at 1 but is not differentiable at 1.
8. Show that the equation $x^7 + 5x + 3 = 0$ has exactly one real solution.
9. (a) Let $M > 0$. Let f be differentiable on an open interval I and such that $|f'(x)| \leq M$ for each x in I. Show that for all x and y in I, we have
$$\big|f(x) - f(y)\big| \leq M|x - y|.$$
   (b) Give an example of a function f defined on the reals with the following properties. There is M such that for all x and y,
$$\big|f(x) - f(y)\big| \leq M|x - y|,$$
   but f is not differentiable everywhere.
10. Let f and g be differentiable on **R**. Assume that for all x in I, $f'(x) \leq g'(x)$.
   (a) Show that if $x \geq 0$, we have
$$f(x) - f(0) \leq g(x) - g(0).$$
   (b) Show that if $x \leq 0$, we have
$$f(x) - f(0) \geq g(x) - g(0).$$
11. Let $r > 1$ be a rational, and $0 \leq x < y$. Show that
$$y^r - x^r < ry^{r-1}(y - x).$$
12. Use the LMVT for $f(x) = \sqrt{1+x}$ to show that if $x > 0$, then
$$\sqrt{1+x} < 1 + x/2.$$
13. Use the LMVT to show that for all x in [0, 1), we have
$$\ln(1 - x) \leq -x.$$
14. Use the LMVT to show that for all x in [0, 1), we have
$$\ln(1 - x) \geq \frac{-x}{1 - x}.$$
15. Let f be differentiable on an open interval I. Let a be in I, and $\delta > 0$ such that $(a - \delta, a + \delta) \subset I$. Assume that if $a < x < a + \delta$, then $f'(x) > 0$ and if $a - \delta < x < a$, then $f'(x) < 0$. Show that f has a local minimum at a.

16. In this problem we prove the second derivative test. Let f and f' be differentiable on an open interval I. Assume that a is in I, $f'(a) = 0$, and $f''(a) > 0$. Since I is open, there is $\delta > 0$ such that $(a - \delta, a + \delta) \subset I$. Define the function ϕ on $(-\delta, \delta)$ by

$$\phi(h) = \frac{f'(a+h) - f'(a)}{h} \quad \text{for } h \neq 0$$

and $\phi(0) = f''(a)$.
(a) Show that ϕ is continuous on $(-\delta, \delta)$.
(b) Show that there is $\delta_1 > 0$ such that $\phi > 0$ on $(-\delta_1, \delta_1)$. (See Application 5.1 in Sect. 5.1.)
(c) Show that for $0 < h < \delta_1$, we have $f'(a+h) > 0$, and for $-\delta_1 < h < 0$, we have $f'(a+h) < 0$.
(d) Conclude that f has a local minimum at a. (Use Exercise 15.)
17. Assume that f is differentiable everywhere except possibly at 0. Assume also that f is continuous everywhere.
(a) If f' has a limit at 0, show that f is differentiable at 0.
(b) Use the function $g(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$ to show that the converse of (a) is not true.
18. Give an example of a function defined on [0, 1], differentiable on (0, 1), and such that the LMVT does not hold.

5.4 Intervals, Continuity, and Inverse Functions[1]

An interval is a subset of **R** that has no holes. More formally:

Intervals
A subset $I \neq \emptyset$ of the reals is said to be an interval if for all $x < y$ in I, any z such that $x \leq z \leq y$ is also in I.

A typical example of interval is a subset of the reals such as

$$\{x \in \mathbf{R} : -2 < x \leq 1\}$$

that is denoted by $(-2, 1]$.

Example 5.28 Let $I \neq \emptyset$ be an interval. Assume that I is bounded above but not below. Show that there is a in **R** such that either $I = (-\infty, a)$ or $I = (-\infty, a]$.

Since I is not empty and is bounded above, it has a least upper bound (by the fundamental property of the reals) that we denote by a. There are two possibilities.

[1]The material in this section may be difficult for a beginner. It is not essential for the sequel and may be omitted.

Either a belongs to I, or it does not. Assume first that a belongs to I. Since a is an upper bound of I if x belongs to I, we have that $x \le a$. Hence, $I \subset (-\infty, a]$. We want to show that in fact $I = (-\infty, a]$. Let $y < a$. Since I is not bounded below, there must be a z in I such that $z < y$. Hence, $z < y < a$ where z and a are in I. Since I is an interval, y is in I. This proves that $(-\infty, a] \subset I$, and so $(-\infty, a] = I$.

The second possibility is that a, the least upper bound of I, is not in I. We have that $I \subset (-\infty, a)$. Let $y < a$. Since a is the least upper bound of I, y cannot be an upper bound of I. Thus, there is z_1 in I such that $y < z_1 < a$. On the other hand, since I is not bounded below, there is a z_2 in I such that $z_2 < y$. Hence, $z_2 < y < z_1$ with z_1 and z_2 in I. Using the definition of an interval, we get that y is in I. Therefore, $(-\infty, a) \subset I$ and $I = (-\infty, a)$.

By using the method of Example 5.28 it is possible to show (see the exercises) that all intervals are of the type above. More precisely, if I is an interval, then either there is a real a such that $I = (-\infty, a)$ or $(-\infty, a]$, or (a, ∞) or $[a, \infty)$, or there are reals a and b such that $I = (a, b)$, $[a, b)$, $(a, b]$, or $[a, b]$. Finally, $\mathbf{R} = (-\infty, +\infty)$ is also an interval.

Intervals such as $(-\infty, a)$, (a, ∞), and (a, b) are said to be open. Intervals such as $(-\infty, a]$, $[a, \infty)$, and $[a, b]$ are said to be closed. Intervals such as $(a, b]$ are neither open nor closed.

Recall that a function f is said to be one-to-one on D if for all x and y in D, $f(x) = f(y)$ implies $x = y$.

The next result makes a link between several notions.

One-to-one and strictly monotone functions

Let f be continuous on an interval I. The function f is one-to-one on I if and only if it is strictly monotone.

One direction is immediate. Assume that f is strictly monotone. Then if $f(x) = f(y)$, we must have $x = y$. For if there is a strict inequality between x and y, there must be a strict inequality between $f(x)$ and $f(y)$ (since f is strictly monotone). Hence, f is one-to-one.

Assume now that f is one-to-one and continuous on the interval I. We want to show that f is strictly monotone. We do a proof by contradiction. If f is not strictly monotone, there must be three reals x, y, z in I with $x < y < z$ such that either $(f(x) < f(y)$ and $f(y) > f(z))$ or $(f(x) > f(y)$ and $f(y) < f(z))$. Both cases will lead to a contradiction. We treat the first one. The second one will be done in the exercises.

Assume that $x < y < z$, $f(x) < f(y)$, and $f(y) > f(z)$. There are two subcases. Either $f(z) < f(x)$ or $f(z) > f(x)$. Consider first $f(z) < f(x)$. Let $c = f(x)$, then $f(z) < c < f(y)$. Since f is continuous on $[y, z]$, by the Intermediate Value Theorem, there is x_1 in (y, z) such that $f(x_1) = c$. But $f(x) = c$ as well, and f is one-to-one. Hence, $x = x_1$. Since $x < y$ and $x_1 > y$, we get a contradiction. We cannot have $f(z) < f(x)$. Since $f(z) \neq f(x)$ (why not?), we must have $f(z) > f(x)$.

Let $d = f(z)$. We have $f(x) < d < f(y)$. By the IVT there is z_1 in (x, y) such that $d = f(z_1)$. Since $d = f(z)$ and f is one-to-one, we have $z = z_1$. Since $z_1 < y < z$, we have a contradiction. Hence, we cannot have $f(x) < f(z)$ either. Therefore, assuming that $x < y < z$, $f(x) < f(y)$, and $f(y) > f(z)$ has led to a contradiction. Similarly, if we assume that $x < y < z$, $f(x) > f(y)$, and $f(y) < f(z)$, we also get a contradiction (see the exercises). Thus, f must be strictly monotone.

The preceding theorem depends crucially on the continuity and interval hypotheses. We illustrate the importance that the function be continuous on an interval in the next example.

Example 5.29 Consider the function g defined on $D = (-\infty, 1) \cup (1, \infty)$ by

$$g(x) = \frac{x}{1-x}.$$

Note that g is continuous on D (why?). Assume that $g(x) = g(y)$. Then

$$\frac{x}{1-x} = \frac{y}{1-y}.$$

Hence, $x - xy = y - xy$ and $x = y$. That is, g is one-to-one. Note now that $g(0) = 0 > g(2) = -2$, but $g(2) = -2 < g(3) = -3/2$. Hence, g is not monotone on D. However, this does not contradict the fact that a one-to-one function which is continuous on an interval is monotone. The set D is not an interval!

We now turn to the continuity of inverse functions.

Inverse functions and continuity
Let f be strictly monotone on an interval I. The function f has an inverse f^{-1} defined on the range of f which is denoted by J. Moreover, f^{-1} is continuous on J.

What is remarkable here is that the inverse function of a function not necessarily continuous is continuous! We will give such an example after the proof.

As argued before, it is clear that a strictly monotone function f is one-to-one. Hence, it has an inverse function f^{-1} defined on the range of f,

$$J = \{f(x) : x \in I\}.$$

We now show that f^{-1} is continuous on J. By contradiction, assume that f^{-1} is not continuous at some b in J. Hence, there must be a sequence b_n in J that converges to b but such that $f^{-1}(b_n)$ does not converge to $f^{-1}(b)$. By taking the negation of convergence we get the following. There is $\epsilon > 0$ such that for all N in $\mathbf{N}$, there is $n \geq N$ such that

$$\left|f^{-1}(b_n) - f^{-1}(b)\right| \geq \epsilon.$$

We construct a subsequence of b_n as follows. Take $N = 1$ above. There is $n_1 \geq 1$ such that $|f^{-1}(b_{n_1}) - f^{-1}(b)| \geq \epsilon$. Next, we use $N = n_1 + 1$ to get $n_2 > n_1$ such that $|f^{-1}(b_{n_2}) - f^{-1}(b)| \geq \epsilon$, and so on. We get a subsequence b_{n_k} such that

$$\left|f^{-1}(b_{n_k}) - f^{-1}(b)\right| \geq \epsilon \quad \text{for all } k \geq 1.$$

That is, for every natural k, we have

$$f^{-1}(b_{n_k}) - f^{-1}(b) \geq \epsilon \text{ or } f^{-1}(b_{n_k}) - f^{-1}(b) \leq -\epsilon.$$

Note that either there are infinitely many k for which the first inequality holds or there are infinitely many k for which the second inequality holds. Assume that the first inequality holds for infinitely many k' (the other possibility is treated in a similar way). Let $b_{n'_k}$ a subsequence of b_{n_k} (and a subsubsequence of b_n) for which we have

$$f^{-1}(b_{n'_k}) - f^{-1}(b) \geq \epsilon \quad \text{for all } k \geq 1.$$

Hence, for all $k \geq 1$,

$$f^{-1}(b_{n'_k}) \geq f^{-1}(b) + \epsilon > f^{-1}(b).$$

Since $f^{-1}(b_{n'_k})$ and $f^{-1}(b)$ are in I (which is the range of f^{-1}) and since I is an interval, $f^{-1}(b) + \epsilon$ is also in I. If f is strictly increasing, we have

$$f\left(f^{-1}(b_{n'_k})\right) \geq f\left(f^{-1}(b) + \epsilon\right) > f\left(f^{-1}(b)\right).$$

Hence,

$$b_{n'_k} \geq f\left(f^{-1}(b) + \epsilon\right) > b.$$

Since $b_{n'_k}$ is a subsequence of b_n, it must converge to b. By letting k go to infinity we have

$$b \geq f\left(f^{-1}(b) + \epsilon\right) > b.$$

That is, $b > b$, a contradiction. If we assume that f is strictly decreasing, we also get a contradiction. Therefore, if b_n converges to b, then $f^{-1}(b_n)$ converges to $f^{-1}(b)$. That is, f^{-1} is continuous at b.

Example 5.30 In this example we show that a discontinuous function may have a continuous inverse. Define the function f on $I = [0, 3]$ by

$$f(x) = x \quad \text{for } 0 \leq x \leq 2 \quad \text{and} \quad f(x) = x + 1 \quad \text{for } 2 < x \leq 3.$$

We start by showing that f is not continuous at 2. Let $a_n = 2 - 1/n$. Then a_n converges to 2. For every $n \geq 1$, a_n is in I, and $a_n < 2$. Hence, $f(a_n) = a_n$, which converges to 2. Define now $b_n = 2 + 1/n$; it also converges to 2 and is in I. Since $b_n > 2$, we have $f(b_n) = b_n + 1$, which converges to 3. Since a_n and b_n converge to 2 but $f(a_n)$ and $f(b_n)$ converge to different limits, f is not continuous at 2.

It is clear that f is strictly increasing on I. Hence, it is one-to-one and has an inverse function f^{-1}. It is also easy to see that the range of f is $J = [0, 2] \cup (3, 4]$ (note that J is not an interval) and that

$$f^{-1}(x) = x \quad \text{for } 0 \leq x \leq 2 \quad \text{and} \quad f^{-1}(x) = x - 1 \quad \text{for } 3 < x \leq 4.$$

We now show that f^{-1} is continuous on J. We start with $x = 2$. Assume that a_n is in J and converges to 2. Take $\epsilon = 1/2 > 0$. There exists a natural N such that

$$|a_n - 2| < 1/2 \quad \text{for all } n \geq N.$$

Hence, $a_n < 5/2$ for every $n \geq N$. Since a_n is also in $J = [0, 2] \cup (3, 4]$, we must have $a_n \leq 2$ for every $n \geq N$. Therefore, $f^{-1}(a_n) = a_n$ for every $n \geq N$, and so $f^{-1}(a_n)$ converges to $2 = f^{-1}(2)$. Hence, f^{-1} is continuous at 2.

Consider now a point b in J different from 2. Either b is in $[0, 2)$ or in $(3, 4]$. Assume that it is in $[0, 2)$ (the other case is similar). Let b_n be a sequence in J that converges to 2. Take $\epsilon = \frac{2-b}{2} > 0$. There exists N in $\mathbf{N}$ such that if $n \geq N$, then

$$|b_n - b| < \epsilon = \frac{2-b}{2}.$$

Therefore,

$$b_n < b + \epsilon = \frac{2+b}{2} < 2.$$

Hence, $f^{-1}(b_n) = b_n$ for $n \geq N$. This sequence converges to b, which is also $f^{-1}(b)$. Thus, f^{-1} is continuous at b. Therefore, even though f is not continuous on I, its inverse function f^{-1} is continuous on J.

As pointed out in the preceding section, given $\alpha \neq 0$, the function f defined by

$$f(x) = x^\alpha = \exp(\alpha \ln x)$$

is continuous and differentiable on $(0, \infty)$. However, these results rely on properties of the exp function that will only be proved in Chap. 7. In the application below we show that if α is rational, then we can prove that this function is continuous. We will also prove below that it is differentiable.

Application 5.2 Let $r > 0$ be a rational. The function f defined on $I = [0, \infty)$ by $f(x) = x^r$ is continuous on I.

Let $r = p/q$ for p and q naturals. We know that the function $g : x \to x^q$ is continuous on I for every natural q. We also know that it is a strictly increasing function and that its inverse function $g^{-1} : x \to x^{1/q}$ is defined on $J = I = [0, \infty)$ (see Sect. 1.4). Since g is strictly monotone on the interval I, we have that g^{-1} is continuous on J. Let $h : x \to x^p$; this is also a continuous function on I. We have

$$f = h \circ g^{-1}.$$

Hence, as a composition of continuous functions, f is continuous at a.

Here is another connection between strict monotonicity, continuity, and intervals.

Continuity, strict monotonicity, and intervals

(a) Let f be continuous on an interval I. Then the range of f

$$J = \{f(x) : x \in I\} \quad \text{is an interval.}$$

(b) Let f be strictly monotone on an interval I. If the range of f is an interval, then f is continuous on I.

We prove (a) first. Assume that the function f is continuous on I. Let $x < y$ in J. Assume that z is strictly between x and y. We want to show that z is in J as well. By the definition of J, there are x_1 and y_1 in I such that $f(x_1) = x$ and $f(y_1) = y$. In particular, $f(x_1) < z < f(y_1)$. Since f is continuous on I, it is continuous on $[x_1, y_1]$. By the intermediate value theorem, there is z_1 in $[x_1, y_1]$ such that $f(z_1) = z$. That is, z is also in J. This proves that J is an interval.

We now turn to the proof of (b). Since f is strictly monotone on I, f has an inverse function f^{-1} defined on the range of f, J. We now show that the function f^{-1} is strictly monotone. Assume that $x < y$ are in J. There are x_1 and y_1 in I such that $x = f(x_1)$ and $y = f(y_1)$. Assume that f is strictly increasing (the decreasing case is similar). By contradiction, assume that

$$x_1 \geq y_1.$$

Since f is increasing, we get

$$f(x_1) \geq f(y_1).$$

That is, $x \geq y$, a contradiction. We must have $x_1 < y_1$. Since $x_1 = f^{-1}(x)$ and $y_1 = f^{-1}(y)$, we get that f^{-1} is strictly increasing on J, which is an interval by assumption. Hence, since the inverse function of a function defined on an interval is continuous, we have that $(f^{-1})^{-1}$ is continuous on I. But $(f^{-1})^{-1} = f$. Hence, f is continuous. This concludes the proof of (b).

Application 5.3 Let f be continuous and strictly monotone on the open interval I. Then, the range of f, J, is an open interval as well.

Since f is continuous, we know that J is an interval. We need to prove that J is open. In order to prove that J is open on the right, it is enough to show that either J is not bounded above, or it is bounded above, but its least upper bound does not belong to J. Assume that J is bounded above by some M. Since J is not empty (intervals are not empty according to our definition), by the fundamental property of the reals, J has a least upper bound b. By contradiction, assume that b belongs to J. Then b is in the range of f, and there is a in I such that $f(a) = b$. Assume that f is strictly increasing (the decreasing case is similar). Since I is open, there is $c > a$ in I, and therefore $f(c) > f(a) = b$. Hence, $f(c)$ is in J and is strictly larger than the upper bound b of J. We have a contradiction. The l.u.b. b does not belong to J, and J is open on the right. Similarly, one can show that it is open on the left as well.

Inverse function and differentiability
Let f be continuous and one-to-one on the open interval I. Assume that f is differentiable at a in I and that $f'(a) \neq 0$. Let $b = f(a)$. Then f^{-1} is differentiable at $b = f(a)$, and

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

We know that J, the range of f, is an interval. In fact, J is an open interval (see Application 5.3). This is important because it tells us that b is not a boundary point of J, and therefore there is an interval centered at b where f^{-1} is defined. Let k_n be a nonzero sequence converging to 0 such that $b+k_n$ is in J for all $n \geq 1$. Let the sequence h_n be defined by

$$h_n = f^{-1}(b+k_n) - a.$$

Note that if $h_n = 0$, then $f^{-1}(b+k_n) = a = f^{-1}(b)$ and $k_n = 0$, which is not possible. Hence, h_n is also a nonzero sequence. Since f^{-1} is continuous (why?), $f^{-1}(b+k_n)$ converges to $f^{-1}(b) = a$. Thus, h_n converges to 0. Since f is differentiable at a,

$$\frac{f(a+h_n) - f(a)}{h_n}$$

converges to $f'(a)$. Note that

$$f(a+h_n) - f(a) = f\left(f^{-1}(b+k_n)\right) - f\left(f^{-1}(b)\right) = b + k_n - b = k_n.$$

Hence,

$$\frac{f(a+h_n) - f(a)}{h_n} = \frac{k_n}{f^{-1}(b+k_n) - f^{-1}(b)},$$

which converges to $f'(a)$. Since $f'(a)$ is not 0, we have that

$$\frac{f^{-1}(b+k_n) - f^{-1}(b)}{k_n}$$

converges to $1/f'(a)$. This shows that f^{-1} is differentiable at b and the stated formula.

Example 5.31 Let n be a natural number. Then the function g defined by

$$g(x) = x^{1/n}$$

is differentiable on $(0, \infty)$. Moreover, $g'(x) = \frac{1}{n}x^{1/n-1}$.

Let f be defined by

$$f(x) = x^n.$$

The function f is differentiable on the open interval $I = (0, \infty)$. Moreover, it is one-to-one, and its inverse function is g (see Sect. 1.4). Since $a > 0$, we have $f'(a) = na^{n-1} > 0$. Hence, we may apply the formula above to get

$$g'(b) = \frac{1}{f'(g(b))} = \frac{1}{n(b^{1/n})^{n-1}} = \frac{1}{n} b^{1/n-1}.$$

The derivative of x^r
Let r be a rational. Then the function f defined by $f(x) = x^r$ is differentiable at any a in $(0, \infty)$. Moreover, $f'(a) = ra^{r-1}$.

We will prove the case $r > 0$ and leave $r \leq 0$ to the exercises. There are natural numbers p and q such that $r = p/q$. Let g and h be defined on $(0, \infty)$ by

$$g(x) = x^{1/q} \quad \text{and} \quad h(x) = x^p.$$

Note that $f(x) = x^r = h(g(x))$. Let $a > 0$. Then $g(a) > 0$. Hence, g is differentiable at a, and h at $g(a)$. Moreover, by the chain rule and Example 5.30 we have

$$f'(a) = h'\big(g(a)\big)g'(a) = pg(a)^{p-1}\frac{1}{q}a^{1/q-1} = r\big(a^{1/q}\big)^{p-1}a^{1/q-1} = ra^{r-1}.$$

Example 5.32 The function ln is differentiable on $(0, \infty)$.

We use the following facts about exp. It is differentiable everywhere, its derivative is itself, and its range is $J = (0, \infty)$. Since the derivative of exp is strictly positive on the interval $\mathbf{R} = (-\infty, \infty)$, we know that it is strictly increasing and therefore one-to-one. Its inverse function ln is defined on the range of exp. For any b in J, there is a unique a in $\mathbf{R}$ such that $b = \exp(a)$. Since exp is never 0, ln is differentiable at b, and

$$(\ln)'(b) = \frac{1}{\exp'(\ln b)} = \frac{1}{b}.$$

Hence, ln is differentiable on J.

Exercises

1. Assume that f is continuous and one-to-one on the interval I. Suppose that $x < y < z$, $f(x) > f(y)$, and $f(y) < f(z)$ for x, y, z in I. Find a contradiction.
2. Give an example of a function which is one-to-one on $D = \{-1, 0, 1\}$ but not monotone.
3. Consider the function sin on $I = (-\pi/2, \pi/2)$. Use the following facts about sin on I. It is continuous, differentiable, and its derivative cos is strictly positive on I.
   (a) Show that sin has an inverse function denoted by arcsin.

(b) Find the interval J where arcsin is defined.
(c) Show that arcsin is differentiable on J and compute its derivative.

4. Consider the function f defined on $D = (-\infty, 0) \cup (0, \infty)$ by $f(x) = 1/x$.
   (a) Show that f is one-to-one on D.
   (b) Show that f is not monotone on D.
   (c) Do (a) and (b) contradict the result that states that a continuous function on an interval is one-to-one if and only if it is strictly monotone?
5. Consider the function f defined on $I = [0, 3]$ by
$$f(x) = x \quad \text{for } 0 \le x \le 2 \quad \text{and} \quad f(x) = x + 1 \quad \text{for } 2 < x \le 3.$$
   (a) Show that f is strictly increasing on I.
   (b) Show that f is one-to-one.
   (c) Show that the range of f is $J = [0, 2] \cup (3, 4]$.
   (d) Show that f^{-1} is defined on J by
$$f^{-1}(x) = x \quad \text{for } 0 \le x \le 2 \quad \text{and} \quad f^{-1}(x) = x - 1 \quad \text{for } 3 < x \le 4.$$
   (e) Let $3 < b \le 4$. Prove that f^{-1} is continuous at b.
6. Draw the graph of a function which is defined on an interval I, whose range is an interval J, but which is not continuous.
7. In this exercise we show that if a function f is continuous on a closed and bounded interval $[a, b]$, then its range J is also a closed and bounded interval $[c, d]$.
   (a) Show that J is an interval.
   (b) Explain why J has a minimum and a maximum.
   (c) Conclude.
8. In Application 5.2 we proved that for every rational $r > 0$, the function $x \to x^r$ is continuous on $[0, +\infty)$. Show that if $r < 0$, the function $x \to x^r$ is continuous on $(0, \infty)$.
9. Assume that $I \neq \emptyset$ is an interval.
   (a) If I is not bounded below nor above, show that I is all of $\mathbf{R}$.
   (b) If I is bounded below but not above, show that there is a real a such that $I = [a, \infty)$ or $I = (a, \infty)$.
   (c) By using the method in (a) and (b) describe all the intervals.
10. Let $r < 0$ be a rational. Show that f defined by $f(x) = x^r$ is differentiable on $(0, \infty)$ and that $f'(a) = ra^{r-1}$ for $a > 0$.
11. Assume that the function f is differentiable on $\mathbf{R}$, its range is $(0, \infty)$, and its derivative is itself.
   (a) Show that f is one-to-one.
   (b) Show that f^{-1} is differentiable on $(0, \infty)$ and find its derivative.
12. Let f be one-to-one and differentiable on the open interval I. Assume that $f'(a) = 0$ for some a in I. Prove that f^{-1} is not differentiable at $f(a)$. (Do a proof by contradiction: assume that f^{-1} is differentiable at $f(a)$ and show that this implies that $(f^{-1})'(f(a)) \times f'(a) = 1$.)
13. Consider the function f defined by $f(x) = \frac{1}{1+x^2}$ for all x in $\mathbf{R}$.
   (a) Show that the range of f is not open.

(b) We know that the range of a continuous strictly monotone function on an open interval is open (see Application 5.3). Which hypothesis does not hold here?

14. Let g be a continuous strictly decreasing function on $[0, 1)$. Show that the range of g is an interval $(a, b]$ for some reals a and b or an interval $(-\infty, b]$.

Chapter 6
Riemann Integration

6.1 Construction of the Integral

We start with some notation. Consider a bounded function f on a closed and bounded interval $[a,b]$. The set P is said to be a partition of $[a,b]$ if there is a natural n such that

$$P = \{x_0, x_1, \ldots, x_n\}$$

where $x_0 = a < x_1 < \cdots < x_n = b$. In words, P is a finite collection of ordered reals in $[a,b]$ that contains a and b.

The function f is said to be bounded on $[a,b]$ if the set

$$\big\{f(x); x \in [a,b]\big\}$$

has a lower bound and upper bound. Since this set is not empty, we may apply the fundamental property of the reals to get the existence of a least upper bound $M(f,[a,b])$ and a greatest lower bound $m(f,[a,b])$. In particular,

$$m\big(f,[a,b]\big) \le f(x) \le M\big(f,[a,b]\big) \quad \text{for all } x \in [a,b].$$

Throughout this chapter we will assume that f is bounded. For a fixed partition $P = \{x_0, x_1, \ldots, x_n\}$ and for all $i = 1, \ldots, n$, the set

$$\big\{f(x); x \in [x_{i-1}, x_i]\big\}$$

has a greatest lower bound $m(f,[x_{i-1},x_i])$ and a least upper bound $M(f,[x_{i-1},x_i])$ (why?). In particular,

$$m\big(f,[x_{i-1},x_i]\big) \le f(x) \le M\big(f,[x_{i-1},x_i]\big) \quad \text{for all } x \in [x_{i-1},x_i].$$

Intuitively the integral of f between a and b is the area between the graph of f and the x axis. See Fig. 6.1. However, the construction of the integral is involved and will require several steps.

We are now ready for our first definition.

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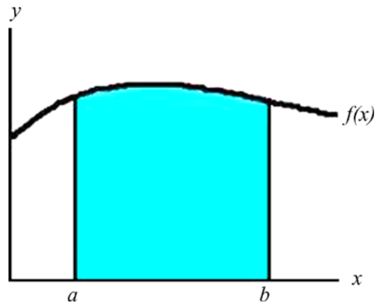

Fig. 6.1 The shaded area represents the integral of the function f between a and b

Lower and Upper Darboux sums

Let f be defined and bounded on the closed and bounded interval $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. The upper Darboux sum corresponding to P is defined by

$$U(f, P) = \sum_{i=1}^{n} M\big(f, [x_{i-1}, x_i]\big)(x_i - x_{i-1}).$$

The lower Darboux sum corresponding to P is defined by

$$L(f, P) = \sum_{i=1}^{n} m\big(f, [x_{i-1}, x_i]\big)(x_i - x_{i-1}).$$

For every partition P, we have

$$L(f, P) \le U(f, P).$$

The inequality $L(f, P) \le U(f, P)$ is a direct consequence of

$$m\big(f, [x_{i-1}, x_i]\big) \le M\big(f, [x_{i-1}, x_i]\big)$$

for $i = 1, \ldots, n$.

The Riemann integral of a function f on $[a, b]$ is the shaded area between the graph of f and the x axis in the graph below. Intuitively, the integral may be computed as a limit of upper or lower Darboux (do not pronounce the 'x' in this name) sums as the number of points in the partition increases to infinity. These limits exist and are equal, provided that the function f is not too irregular. In fact, it can be shown that this is the case if and only if the set of x where f is not continuous is negligible (in a sense to be made precise). See, for instance, the Riemann–Lebesgue theorem in 'An introduction to analysis' by J.R. Kirkwood, second edition, PWS Publishing Company.

Example 6.1 Consider the function f defined on $[0, 1]$ by $f(x) = x^2$. Let P be the partition $\{0, 1/2, 1\}$. This is an increasing function, and we can compute easily

$$m\big(f, [0, 1/2]\big) = 0, \qquad m\big(f, [1/2, 1]\big) = 1/4,$$
$$M\big(f, [0, 1/2]\big) = 1/4, \qquad M\big(f, [1/2, 1]\big) = 1.$$

This yields

$$L(f, P) = 0 \times 1/2 + 1/4 \times 1/2 = 1/8$$

and

$$U(f, P) = 1/4 \times 1/2 + 1 \times 1/2 = 5/8.$$

With more points in our partition, we would get, of course, a better approximation of the area between the x axis and the graph of f. See the exercises.

We are now ready to define Riemann integrability.

Riemann integrability
Let f be defined and bounded on the closed and bounded interval $[a, b]$. The function f is Riemann integrable if for every $\epsilon > 0$, it is possible to find a partition P of $[a, b]$ such that

$$0 \le U(f, P) - L(f, P) < \epsilon.$$

Note that this defines only the notion of integrability, not the integral (which will be defined later on). The notion of integrability relates to the regularity of the function, it is similar and related to continuity and differentiability.

In the examples below we will use several times the following lemma on telescoping sums.

Lemma 6.1 *Let a_n be a sequence of reals. For every $n \ge 1$, we have*

$$\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0.$$

We do an induction on n. We have

$$\sum_{k=1}^{1} (a_k - a_{k-1}) = a_1 - a_0,$$

and the formula holds for $n = 1$. Assume that it holds for n. Splitting the following sum into two, we get

$$\sum_{k=1}^{n+1} (a_k - a_{k-1}) = \sum_{k=1}^{n} (a_k - a_{k-1}) + (a_{n+1} - a_n).$$

Using the induction hypothesis, we have

$$\sum_{k=1}^{n+1}(a_k - a_{k-1}) = (a_n - a_0) + (a_{n+1} - a_n) = a_{n+1} - a_0.$$

This proves Lemma 6.1.

Example 6.2 Constant functions on $[a, b]$ are Riemann integrable.

Let $f(x) = c$ for all x in $[a, b]$. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Since f is constant, it is clear that for all $i = 1, \dots, n$, we have

$$m\big(f, [x_{i-1}, x_i]\big) = M\big(f, [x_{i-1}, x_i]\big) = c.$$

Hence,

$$U(f, P) = L(f, P) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(x_n - x_0) = c(b - a)$$

by Lemma 6.1. In particular, for any $\epsilon > 0$,

$$0 = U(f, P) - L(f, P) < \epsilon.$$

The constant function f is integrable.

Example 6.3 Let f be defined on $[0, 1]$ by $f(x) = 1$ if x is rational and $f(x) = 0$ if f is irrational. Then f is not Riemann integrable on $[0, 1]$.

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[0, 1]$. For $i = 1, \dots, n$, we have rationals and irrationals in $[x_{i-1}, x_i]$ (recall that the rationals and the irrationals are dense in the reals). Hence, f takes the values 0 and 1 in the set $[x_{i-1}, x_i]$. In particular,

$$m\big(f, [x_{i-1}, x_i]\big) = 0 \quad \text{and} \quad M\big(f, [x_{i-1}, x_i]\big) = 1$$

for all $i = 1, \dots, n$. Therefore,

$$U(f, P) = \sum_{i=1}^{n} 1(x_i - x_{i-1}) = x_n - x_0 = 1 - 0 = 1$$

by Lemma 6.1. On the other hand,

$$L(f, P) = \sum_{i=1}^{n} 0(x_i - x_{i-1}) = 0$$

for every partition P. Hence, for every partition P,

$$U(f, P) - L(f, P) = 1.$$

In particular, for $\epsilon = 1/2$, there is no P such that $U(f, P) - L(f, P) < \epsilon$. Therefore, f is not Riemann integrable.

Note that in both of the preceding examples, $U(f, P)$ and $L(f, P)$ do not depend on P. This is due to the very particular examples we took. In general, they will depend on P.

Monotone functions are integrable
Let f be defined, bounded, and monotone on $[a, b]$. Then the function f is Riemann integrable.

Let $\epsilon > 0$. We can pick n so that

$$\frac{b-a}{n}\big(f(b) - f(a)\big) < \epsilon$$

(why?). Then we define the partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ by setting

$$x_i = a + i\frac{b-a}{n} \quad \text{for } i = 0, 1, \ldots, n.$$

Assume that f is increasing on $[a, b]$ (the decreasing case is analogous and is left to the exercises). Then

$$m\big(f, [x_{i-1}, x_i]\big) = f(x_{i-1}) \quad \text{and} \quad M\big(f, [x_{i-1}, x_i]\big) = f(x_i).$$

Hence,

$$U(f, P) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \quad \text{and} \quad L(f, P) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Using that $x_i - x_{i-1} = (b-a)/n$ for every $i = 1, \ldots, n$, we have

$$U(f, P) = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i) \quad \text{and} \quad L(f, P) = \frac{b-a}{n}\sum_{i=1}^{n} f(x_{i-1}).$$

By Lemma 6.1,

$$U(f, P) - L(f, P) = \frac{b-a}{n}\sum_{i=1}^{n}\big(f(x_i) - f(x_{i-1})\big) = \frac{b-a}{n}\big(f(x_n) - f(x_0)\big).$$

That is,

$$U(f, P) - L(f, P) = \frac{b-a}{n}\big(f(b) - f(a)\big).$$

But this is less than ϵ by our choice of n. This proves that f is Riemann integrable.

Another large class of Riemann integrable functions are the continuous functions. However, in order to prove this, we will need the notion of uniform continuity that we now introduce.

Uniformly continuous functions
A function f is said to be uniformly continuous on the set D if for every $\epsilon > 0$, there is a $\delta > 0$ such that if x and y are in D and if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Recall that a function f is said to be continuous on D if for every $a \in D$ and for every $\epsilon > 0$, there is a $\delta > 0$ such that if x is in D and if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. There are some subtle but crucial differences between the two definitions. First, continuity concerns points (f is continuous at every a in D), and uniform continuity concerns sets (f is uniformly continuous on the whole set D). Second, if f is continuous at a, for a given ϵ, the corresponding δ may depend on a and on ϵ. For a different point b in D, the corresponding δ will typically be different. It may or may not be possible to pick the same δ (for a given ϵ) for all the points of a set D. If this is possible, the function is said to be uniformly continuous.

Continuous functions are Riemann integrable
Let f be continuous on $[a, b]$. Then f is Riemann integrable on $[a, b]$.

A continuous function on a closed and bounded interval $[a, b]$ is uniformly continuous on $[a, b]$. This will be proved in Sect. 6.3 whose focus is uniform continuity. Let $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$, there is $\delta > 0$ such that if x and y are in $[a, b]$ and $|x - y| < \delta$, then

$$\big|f(x) - f(y)\big| < \frac{\epsilon}{b - a}.$$

By the Archimedean property we may find n so that

$$\frac{b - a}{n} < \delta.$$

Now that n is fixed, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ defined by

$$x_i = a + i\frac{b - a}{n} \quad \text{for } i = 0, 1, \ldots, n.$$

For any $i = 1, \ldots, n$, f is continuous on $[x_{i-1}, x_i]$, and the extreme value theorem states that f attains its maximum and minimum on this interval. Thus, there are c_i and d_i in $[x_{i-1}, x_i]$ such that

$$m\big(f, [x_{i-1}, x_i]\big) = f(c_i) \quad \text{and} \quad M\big(f, [x_{i-1}, x_i]\big) = f(d_i).$$

Hence,

$$U(f, P) = \sum_{i=1}^{n} f(d_i)(x_i - x_{i-1}) \quad \text{and} \quad L(f, P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

Using that for every $i = 1, \ldots, n$, we have $x_i - x_{i-1} = (b - a)/n$, it follows that

$$U(f, P) = \frac{b - a}{n}\sum_{i=1}^{n} f(d_i) \quad \text{and} \quad L(f, P) = \frac{b - a}{n}\sum_{i=1}^{n} f(c_i).$$

Therefore,

$$U(f, P) - L(f, P) = \frac{b - a}{n}\sum_{i=1}^{n}\big(f(d_i) - f(c_i)\big).$$

Note that since for every $i = 1, \ldots, n$, c_i and d_i are in $[x_{i-1}, x_i]$, we have

$$|c_i - d_i| \le |x_{i-1} - x_i| = \frac{b-a}{n} < \delta$$

by our choice of n. Hence,

$$0 \le f(d_i) - f(c_i) < \frac{\epsilon}{b-a}$$

and

$$U(f, P) - L(f, P) = \frac{b-a}{n} \sum_{i=1}^{n} \big(f(d_i) - f(c_i)\big) \le \frac{b-a}{n} \sum_{i=1}^{n} \frac{\epsilon}{b-a} = \epsilon.$$

This proves that f is Riemann integrable.

Up to this point we have not defined the integral of a function. We now do so.

The Riemann integral

Let f be bounded on $[a, b]$. Then the set of upper Darboux sums has the greatest lower bound $I(U, f, [a, b])$, and the set of lower Darboux sums has the least upper bound $I(L, f, [a, b])$. If f is integrable, then $I(U, f, [a, b]) = I(L, f, [a, b])$, and this real number is denoted by

$$\int_a^b f.$$

Moreover, for every partition P of $[a, b]$, we have

$$L(f, P) \le \int_a^b f \le U(f, P).$$

There are several claims to be proved in the definition above. We will prove them at the end of this section.

Example 6.4 Let f be a constant c. Then $\int_a^b f = c(b-a)$.

We already know that a constant function is Riemann integrable. Moreover, for every partition P of $[a, b]$, we have $L(f, P) = U(f, P) = c(b-a)$. Since $\int_a^b f$ is between the lower and upper sums, we must have $\int_a^b f = c(b-a)$.

Fundamental Theorem of Calculus

Let f be Riemann integrable on $[a, b]$, and let F be continuous on $[a, b]$ and differentiable on (a, b). Assume that $F'(x) = f(x)$ for all x in (a, b). Then

$$\int_a^b f = F(b) - F(a).$$

We now prove the FTC. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Using that $x_0 = a$ and $x_n = b$ and Lemma 6.1, we get

$$F(b) - F(a) = F(x_n) - F(x_0) = \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big).$$

Since F is continuous on $[x_{i-1}, x_i]$ and differentiable on (x_{i-1}, x_i), for $i = 1, \dots, n$, the Lagrange mean value theorem applies: there is y_i in (x_{i-1}, x_i) such that

$$F(x_i) - F(x_{i-1}) = F'(y_i)(x_i - x_{i-1}) = f(y_i)(x_i - x_{i-1}).$$

In particular,

$$F(b) - F(a) = \sum_{i=1}^{n} f(y_i)(x_i - x_{i-1}).$$

Observe that for every $i = 1, \dots, n$, we have

$$m\big(f, [x_{i-1}, x_i]\big) \le f(y_i) \le M\big(f, [x_{i-1}, x_i]\big).$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n} m\big(f, [x_{i-1}, x_i]\big)(x_i - x_{i-1}) &\le \sum_{i=1}^{n} f(y_i)(x_i - x_{i-1})\\ &\le \sum_{i=1}^{n} M\big(f, [x_{i-1}, x_i]\big)(x_i - x_{i-1}).\end{aligned}$$

That is, for every partition P, we have

$$L(f, P) \le F(b) - F(a) \le U(f, P).$$

Therefore, $F(b) - F(a)$ is an upper bound of the set of lower Darboux sums. Since f is integrable, we know that $\int_a^b f$ is the least upper bound of the set of lower Darboux sums. Thus,

$$\int_a^b f \le F(b) - F(a).$$

On the other hand, $F(b) - F(a)$ is also a lower bound of the set of upper Darboux sums. Since $\int_a^b f$ is also the greatest lower bound of that set, we have

$$F(b) - F(a) \le \int_a^b f.$$

Thus, $\int_a^b f = F(b) - F(a)$, and this completes the proof of the FTC.

Example 6.5 Let $a \ne -1$. Show that f defined by $f(x) = x^a$ on $[1, 2]$ is integrable and find its integral.

Note that $f(x) = \exp(a \ln x)$ is continuous on $[1, 2]$ (as a composition of continuous functions) and is therefore integrable on $[1, 2]$. Let F be defined by

$$F(x) = \frac{1}{a+1} x^{a+1}.$$

Then F is continuous on $[1,2]$ and differentiable on $(1,2)$ with $F' = f$. Hence, the FTC applies, and we have

$$\int_1^2 f = \frac{1}{a+1}\left(2^{a+1} - 1\right).$$

The FTC is very useful when one can find an antiderivative F for a given function f. Unfortunately, there are very few functions for which one can find an antiderivative. The examples from Calculus are, most of the time, manufactured so that the FTC can be applied....

We now turn our attention to the claims made to define the integral of an integrable function.[1] There are four such claims:

1. The set of upper Darboux sums has a greatest lower bound $I(U, f, [a,b])$.
2. The set of lower Darboux sums has a least upper bound $I(L, f, [a,b])$.
3. If f is integrable, then $I(U, f, [a,b]) = I(L, f, [a,b])$.
4. If f is integrable, then for every partition P, we have

$$L(f, P) \le \int_a^b f \le U(f, P).$$

The fourth claim is an easy consequence of the fact that $\int_a^b f$ is a lower bound of upper Darboux sums and an upper bound of lower Darboux sums.

The first three claims will be proved at once thanks to the following lemma. We designate the least upper bound of a set A by $\sup A$ and the greatest lower bound by $\inf A$, if they exist!

Lemma 6.2 *Let A and B be two nonempty subsets of reals with the property that for all x in A and y in B, we have $x \le y$. Then* $\sup A$ *and* $\inf B$ *exist and* $\sup A \le \inf B$. *If, in addition, for every $\epsilon > 0$, there are x in A and y in B such that*

$$y - x < \epsilon,$$

then $\sup A = \inf B$.

We first apply Lemma 6.2, and we then prove it. In our application of this lemma, A is the set of lower Darboux sums, and B is the set of upper Darboux sums. We will show that any lower Darboux sum is less than any upper Darboux sum. According to Lemma 6.2, this implies that A has a least upper bound (denoted by $\sup A$) and B has a greatest lower bound denoted by $\inf B$. Moreover, by the definition of an integrable function we know that for every $\epsilon > 0$, there is a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

Therefore, the second part of Lemma 6.2 applies as well, and we can conclude that $\sup A = \inf B$. That is, $I(U, f, [a,b]) = I(L, f, [a,b])$.

[1] The material in the rest of this section may be difficult for a beginner. It is not essential for the sequel and may be omitted.

Hence, to apply Lemma 6.2, we only need to check that any lower Darboux sum is less than any upper Darboux sum. We start with the so-called refinement lemma.

Lemma 6.3 *Let P and Q be partitions of $[a, b]$ with $P \subset Q$. Then*

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P).$$

The partition Q is said to be a refinement of partition P because in addition to the points of P, Q has other points as well.

We will prove that $L(f, P) \le L(f, Q)$. The proof that $U(f, P) \ge U(f, Q)$ is similar and is omitted. We have already observed that $L(f, Q) \le U(f, Q)$.

We do a proof by induction. Our induction hypothesis is: if Q has k more points than P, then $L(f, P) \le L(f, Q)$. Assume first that $k = 1$. That is, Q has just one more point than P. Let $P = \{x_0, x_1, \ldots, x_n\}$, and let y be the additional point of Q. Assume that y is in (x_{j-1}, x_j) for some j in $\{1, \ldots, n\}$. Note that

$$L(f, P) = \sum_{i=1}^{n} m\big(f, [x_{i-1}, x_i]\big)(x_i - x_{i-1})$$

and

$$\begin{aligned} L(f, Q) = &\sum_{i=1}^{j-1} m\big(f, [x_{i-1}, x_i]\big)(x_i - x_{i-1}) \\ &+ m\big(f, [x_{j-1}, y]\big)(y - x_{j-1}) + m\big(f, [y, x_j]\big)(x_j - y) \\ &+ \sum_{i=j+1}^{n} m\big(f, [x_{i-1}, x_i]\big)(x_i - x_{i-1}). \end{aligned}$$

Hence,

$$\begin{aligned} L(f, Q) - L(f, P) = &\, m\big(f, [x_{j-1}, y]\big)(y - x_{j-1}) + m\big(f, [y, x_j]\big)(x_j - y) \\ &- m\big(f, [x_{j-1}, x_j]\big)(x_j - x_{j-1}). \end{aligned}$$

By writing $x_j - x_{j-1} = x_j - y + y - x_{j-1}$ we get

$$\begin{aligned} L(f, Q) - L(f, P) = &\Big(m\big(f, [x_{j-1}, y]\big) - m\big(f, [x_{j-1}, x_j]\big)\Big)(y - x_{j-1}) \\ &+ \Big(m\big(f, [y, x_j]\big) - m\big(f, [x_{j-1}, x_j]\big)\Big)(x_j - y). \end{aligned}$$

It is easy to see that if $C \subset D$, then $\inf C \ge \inf D$ (see the exercises). Since

$$\big\{f(x), x \in [x_{j-1}, y]\big\} \subset \big\{f(x), x \in [x_{j-1}, x_j]\big\},$$

we have

$$m\big(f, [x_{j-1}, y]\big) \ge m\big(f, [x_{j-1}, x_j]\big).$$

Similarly,

$$m\big(f, [y, x_j]\big) \ge m\big(f, [x_{j-1}, x_j]\big).$$

Hence, $L(f, Q) - L(f, P)$ is the sum of two positive terms and is therefore positive. This proves the induction hypothesis for $k = 1$.

Assume now that the induction hypothesis holds for k. Let Q have $k + 1$ more points than P. Let R be the partition obtained by deleting one of the additional points of Q. In particular, Q has one more point than R, and so

$$L(f, R) \leq L(f, Q).$$

On the other hand, R has k more points than P, and by the induction hypothesis we have

$$L(f, P) \leq L(f, R).$$

Putting together the two inequalities above, we get

$$L(f, P) \leq L(f, Q),$$

and the lemma is proved by induction.

Lemma 6.3 is the main ingredient to prove the following:

Lemma 6.4 *Let P and Q be any two partitions of* $[a, b]$. *Then*

$$L(f, P) \leq U(f, Q).$$

Define R as the union of P and Q. Hence, R is a refinement of both P and Q. Using Lemma 6.3 to get the first and third inequalities below, we get

$$L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, Q).$$

This completes the proof of Lemma 6.4.

With Lemma 6.4 in hand, Lemma 6.2 implies that

$$I\big(U, f, [a, b]\big) = I\big(L, f, [a, b]\big).$$

Our last task in this section is the proof of Lemma 6.2.

Recall that we are assuming that A and B are two nonempty subsets of reals with the property that for all x in A and y in B, we have $x \leq y$. Fix x_0 in A and y_0 in B. Then, for all y in B, we have $y \geq x_0$. That is, B is bounded below and nonempty. Therefore, by the fundamental property of the reals, B has a greatest lower bound $\inf B$. Similarly, A is bounded above by x_0 and is not empty. Hence, it has a least upper bound $\sup A$. Since y_0 is an upper bound of A, we have $y_0 \geq \sup A$. But this is true for any y_0 in B. Hence, $\sup A$ is a lower bound of B. It must be smaller than the greatest lower bound $\inf A$. Thus, $\sup A \leq \inf B$. That proves the first part of Lemma 6.2.

Assume in addition that for every $\epsilon > 0$, we have x in A and y in B such that

$$0 \leq y - x < \epsilon.$$

By the definition of $\sup A$ and $\inf B$ we have

$$x \leq \sup A \quad \text{and} \quad y \geq \inf B.$$

Hence,

$$y - x \geq \inf B - \sup A \geq 0,$$

and for every $\epsilon > 0$, we have

$$0 \leq \inf B - \sup A < \epsilon.$$

Therefore, $\inf B - \sup A$ is necessarily 0 (why?), and we are done.

Exercises

1. Let A be a set of reals bounded below and above. Let B be a nonempty subset of A. Then,
$$\sup A \geq \sup B \quad \text{and} \quad \inf A \leq \inf B.$$
2. Pick a partition of $[0, 1]$ and compute an upper and a lower Darboux sum for the function $f(x) = x$.
3. Redo Example 6.1 with the partition $P = \{0, 1/4, 1/2, 3/4, 1\}$.
4. Let f be an integrable function on $[a, b]$. Assume that there is a real I such that for every partition P,
$$L(f, P) \leq I \leq U(f, P).$$
Show that $I = \int_a^b f$.
5. Draw a triangle on the plane. Use the FTC to compute the area of the triangle. Check your computation with a geometric formula.
6. Justify the existence and compute the following integrals.
   (a) $\int_0^1 \frac{1}{1+x^2}$.
   (b) $\int_0^1 \frac{x}{1+x^2}$.
   (c) $\int_{-2}^2 x \exp(-x^2)$.
   (d) $\int_0^1 \sqrt{x}$.
7. Let f be continuous on $(0, 1]$
   (a) Show that f is integrable on $[1/n, 1]$ for every natural n.
   (b) Give an example of f for which $\int_{1/n}^1 f$ converges as n goes to infinity.
   (c) Give an example of f for which $\int_{1/n}^1 f$ does not converge.
8. Let f be a bounded function on $[a, b]$.
   (a) Show that if f is constant on $[a, b]$, then $U(f, P) = L(f, P)$ for every partition P.
   (b) Show that if there is a partition P of $[a, b]$ such that $U(f, P) = L(f, P)$, then f is constant on $[a, b]$.
9. Show that if f is decreasing on $[a, b]$, then f is integrable.
10. Show that if f is integrable on $[a, b]$ and $a < c < b$, then f is integrable on $[a, c]$.
11. Let $a < c < b$. Assume that f is integrable on $[a, c]$ and on $[c, b]$.
   (a) Let $\epsilon > 0$. Show that there are partitions P and Q of $[a, c]$ and $[c, b]$ such that
$$U(f, P) - L(f, P) < \epsilon/2 \quad \text{and} \quad U(f, Q) - L(f, Q) < \epsilon/2.$$

(b) Let R be the union of P and Q. Show that

$$U(f, R) = U(f, P) + U(f, Q) \quad \text{and} \quad L(f, R) = L(f, P) + L(f, Q).$$

(c) Show that $U(f, R) - L(f, R) < \epsilon$. Conclude that f is integrable on $[a, b]$.

12. Let f be continuous on $(a, b]$ and bounded on $[a, b]$. That is, there is K such that $|f(x)| < K$ for all x in $[a, b]$. In this problem we show that f is integrable.
    (a) Sketch the graph of a function defined on $[a, b]$ but continuous only on $(a, b]$.
    (b) Let $\epsilon > 0$. Show that there is a natural n such that $2K/n < \epsilon/2$.
    (c) Show that there is a partition P of $[a + 1/n, b]$ such that

$$U(f, P) - L(f, P) < \epsilon/2.$$

    (d) Let Q be the partition of $[a, b]$ obtained by adding a to the partition P. Show that

$$\big|U(f, P) - U(f, Q)\big| < K/n \quad \text{and} \quad \big|L(f, P) - L(f, Q)\big| < K/n.$$

    (e) Show that $U(f, Q) - L(f, Q) < \epsilon$. Conclude that f is integrable on $[a, b]$.
13. Let f be bounded on $[a, b]$ and continuous except at one point $c \in (a, b)$.
    (a) Show that f is integrable on $[a, c]$ and on $[c, b]$. Use Exercise 12.
    (b) Show that f is integrable on $[a, b]$. Use Exercise 11.
    (c) Show that if a function is bounded on $[a, b]$ and continuous except at finitely many points, then the function is integrable.
14. Assume that f is 0 on $[a, b]$ except at c where $a < c < b$.
    (a) Use Exercise 13(c) to show that f is integrable.
    (b) Assume that $f(c) > 0$. Show that for every natural n, we have

$$0 \le \int_a^b f \le f(c)\frac{b - a}{n}.$$

    (Recall that for any partition P, we have $L(f, P) \le \int_a^b f \le U(f, P)$.)
    (c) Show that $\int_a^b f = 0$.
15. Show that if a function is 0 except at finitely many points, then the function is integrable, and its integral is 0. (Use Exercise 14.)
16. (a) Assume that f is integrable on $[a, b]$ and that $f = g$ except at finitely many points. Show that g is integrable. (Use Exercise 15.)
    (b) Show that $\int_a^b g = \int_a^b f$.

6.2 Properties of the Integral

We start by stating some useful properties of greatest lower bounds and least upper bounds. Recall that if f is a bounded function on $[a, b]$, then the set

$$\{f(x); x \in [a, b]\}$$

has a least upper bound $M(f, [a, b])$ and a greatest lower bound $m(f, [a, b])$. When there is no ambiguity, we will use the shorter notation $M(f)$ and $m(f)$.

Lemma 6.5 *Let f and g be bounded on $[a,b]$.*

(i) *If $c>0$ is a constant, we have*

$$M\big(cf,[a,b]\big)=cM\big(f,[a,b]\big).$$

(ii) *If $c<0$, then*

$$M\big(cf,[a,b]\big)=cm\big(f,[a,b]\big).$$

(iii) *$f+g$ is also bounded, and*

$$M\big(f+g,[a,b]\big)\le M\big(f,[a,b]\big)+M\big(g,[a,b]\big).$$

(iv) *We also have*

$$m\big(f+g,[a,b]\big)\ge m\big(f,[a,b]\big)+m\big(g,[a,b]\big).$$

We now prove these properties. Since $M(f)$ is an upper bound, we have

$$f(x)\le M(f)\quad \text{for all } x\in[a,b].$$

Assume that $c>0$. Then

$$cf(x)\le cM(f).$$

That is, $cM(f)$ is an upper bound of the set $\{cf(x); x\in[a,b]\}$. Since this set is not empty and bounded above by $cM(f)$, it has a least upper bound $M(cf)$. Hence,

$$M(cf)\le cM(f).$$

The inequality above is true for any bounded function and any positive constant. We use it for cf (instead of f) and $1/c$ (instead of c). This yields

$$M\left(\frac{1}{c}cf\right)\le\frac{1}{c}M(cf).$$

That is, $cM(f)\le M(cf)$. Since we proved already that $cM(f)\ge M(cf)$, we get (i).

To prove (ii), we first consider the particular case $c=-1$. We have

$$f(x)\le M(f)\quad \text{for all } x\in[a,b].$$

Therefore, $-f(x)\ge -M(f)$ for all x in $[a,b]$. So the set

$$\big\{-f(x); x\in[a,b]\big\}$$

is not empty and bounded below by $-M(f)$. Hence, it has a greatest lower bound $m(-f)$, and

$$m(-f)\ge -M(f).$$

On the other hand, we have that

$$f(x)\ge m(f)\quad \text{for all } x\in[a,b].$$

Therefore, $-m(f)$ is an upper bound of $\{-f(x); x\in[a,b]\}$. This set has a least upper bound $M(-f)$ (why?), and so

$$M(-f)\le -m(f).$$

Since this inequality is true for any bounded f, we may replace f by $-f$ in it. We get

$$M\big(-(-f)\big) \leq -m(-f).$$

That is, $M(f) \leq -m(-f)$. But we have already shown that $m(-f) \geq -M(f)$. Hence, $m(-f) = -M(f)$.

Assume now that $c < 0$. Then $c = -|c|$. We have

$$m(cf) = m\big(-|c|f\big) = -M\big(|c|f\big) = -|c|M(f) = cM(f),$$

where the second equality uses the case $c = -1$ (just proved), and the third equality is (i). The equality $m(cf) = cM(f)$ is equivalent to $M(cf) = cm(f)$ (see the exercises), and this proves (ii).

We now turn to (iii). By definition of $M(f)$ and $M(g)$ we have, for all x in $[a, b]$,

$$f(x) \leq M(f) \quad \text{and} \quad g(x) \leq M(g).$$

Hence,

$$f(x) + g(x) \leq M(f) + M(g).$$

That is, $M(f) + M(g)$ is an upper bound of $\{f(x) + g(x); x \in [a, b]\}$. This set has a least upper bound $M(f + g)$ (why?), and so

$$M(f + g) \leq M(f) + M(g).$$

This proves (iii).

The proof of (iv) can be done in a fashion similar to (iii) and is left to the exercises.

We are now ready to state the first properties of the integral.

The integral is linear

Let f be Riemann integrable on $[a, b]$. Let c be a constant. Then cf is integrable, and

$$\text{(S1)} \qquad \int_a^b (cf) = c\int_a^b f.$$

Assume that f and g are integrable on $[a, b]$. Then $f + g$ is integrable, and

$$\text{(S2)} \qquad \int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

We prove S1. Assume first that $c > 0$. Recall that $U(f, P)$ and $L(f, P)$ are the lower and upper Darboux sums for a function f and a partition P. By the definition of integrability, for any $\epsilon > 0$, there is a partition $Q = \{y_0, y_1, \ldots, y_n\}$ such that

$$0 \leq U(f, Q) - L(f, Q) < \epsilon/c.$$

We have

$$U(cf, Q) = \sum_{i=1}^{n} M\big(cf, [y_{i-1}, y_i]\big)(y_i - y_{i-1}).$$

By Lemma 6.5(i),

$$M\big(cf, [y_{i-1}, y_i]\big) = cM\big(f, [y_{i-1}, y_i]\big).$$

Hence,

$$U(cf, Q) = cU(f, Q),$$

and similarly,

$$L(cf, Q) = cL(f, Q).$$

Therefore,

$$0 \le U(cf, Q) - L(cf, Q) = c\big(U(f, Q) - L(f, Q)\big) < c\epsilon/c = \epsilon.$$

Thus, cf is integrable.

Recall that for any integrable function f and any partition Q, we have

$$L(f, Q) \le \int_a^b f \le U(f, Q).$$

We multiply by c to get

$$cL(f, Q) = L(cf, Q) \le c\int_a^b f \le cU(f, Q) = U(cf, Q).$$

Since we know that cf is integrable, we also have

$$L(cf, Q) \le \int_a^b (cf) \le U(cf, Q).$$

Hence,

$$L(cf, Q) - U(cf, Q) \le c\int_a^b f - \int_a^b (cf) \le U(cf, Q) - L(cf, Q).$$

By the definition of the partition Q we have

$$0 \le U(cf, Q) - L(cf, Q) < \epsilon.$$

Therefore,

$$-\epsilon < c\int_a^b f - \int_a^b (cf) < \epsilon.$$

Since this is true for an arbitrarily small $\epsilon > 0$, we must have

$$\int_a^b f - \int_a^b (cf) = 0.$$

This proves S1 for $c > 0$. The proof for $c < 0$ is rather similar. We indicate the necessary steps in the exercises.

We now turn to the proof of S2. By the definition of integrability, for every $\epsilon > 0$, there are partitions P_1 and P_2 of $[a, b]$ such that

$$0 \le U(f, P_1) - L(f, P_1) < \epsilon/2 \quad \text{and} \quad \le U(g, P_2) - L(g, P_2) < \epsilon/2.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be the union of P_1 and P_2. By the refinement lemma (Lemma 6.3 in Sect. 6.1), the inequalities above still hold if we replace P_1 and P_2 by P. On the other hand, by Lemma 6.5(iii), for $i = 1, \ldots, n$,

$$M\big(f + g, [x_{i-1}, x_i]\big) \le M\big(f, [x_{i-1}, x_i]\big) + M\big(g, [x_{i-1}, x_i]\big).$$

Hence,

$$U(f + g, P) = \sum_{i=1}^{n} M\big(f + g, [x_{i-1}, x_i]\big)(x_i - x_{i-1}) \le U(f, P) + U(g, P).$$

Similarly, by Lemma 6.5(iv) we have

$$L(f + g, P) \ge L(f, P) + L(g, P).$$

Therefore,

$$\begin{aligned} 0 &\le U(f + g, P) - L(f + g, P) \le U(f, P) + U(g, P) - L(f, P) - L(g, P) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence, $f + g$ is integrable.

Recall that for every partition P and every integrable function f, we have

$$L(f, P) \le \int_a^b f \le U(f, P).$$

We use the P defined above to get

$$0 \le U(f, P) - \int_a^b f \le U(f, P) - L(f, P) < \epsilon/2.$$

Hence,

$$U(f, P) < \int_a^b f + \epsilon/2.$$

By our choice of P this inequality holds for g as well. Therefore,

$$\int_a^b (f + g) \le U(f + g, P) \le U(f, P) + U(g, P) < \int_a^b f + \epsilon/2 + \int_a^b g + \epsilon/2.$$

That is, for any $\epsilon > 0$, we have

$$\int_a^b (f + g) < \int_a^b f + \int_a^b g + \epsilon.$$

Since $\epsilon > 0$ can be taken arbitrarily small, we must have

$$\int_a^b (f + g) \le \int_a^b f + \int_a^b g.$$

Since this is true for any integrable functions f and g, this must also be true for $-f$ and $-g$. Hence,

$$\int_a^b (-f-g) \le \int_a^b (-f) + \int_a^b (-g).$$

Thus, by S1,

$$-\int_a^b (f+g) \le -\int_a^b f - \int_a^b g,$$

and therefore,

$$\int_a^b (f+g) \ge \int_a^b f + \int_a^b g.$$

Since the reverse inequality also holds, we must have equality, and this concludes the proof of S2.

The integral is additive
Assume that f is integrable on $[a, b]$ and that $a < c < b$. Then f is integrable on $[a, c]$ and $[c, b]$. Moreover,

$$\text{(S3)} \qquad \int_a^b f = \int_a^c f + \int_c^b f.$$

Suppose that f is integrable on $[a, b]$. For any $\epsilon > 0$, there is a partition P of $[a, b]$ for which

$$0 \le U(f, P) - L(f, P) < \epsilon.$$

Let Q be the partition P to which we add the point c (if c is already in P, we take $Q = P$). Since Q is a refinement of P, the inequality above holds for Q as well, see Lemma 6.3 in Sect. 6.1. Let $\{x_0, x_1, \ldots, x_n\}$ be an enumeration of the elements of Q. Since c is in Q, there is a k such that $1 < k < n$ and $x_k = c$. Define $P_1 = \{x_0, x_1, \ldots, x_k\}$ and observe that this is a partition of $[a, c]$. We have

$$\begin{aligned}
&U(f, Q) - L(f, Q) \\
&= \sum_{j=1}^{n} \big(M\big(f, [x_j - x_{j-1}]\big) - m\big(f, [x_j - x_{j-1}]\big)\big)(x_j - x_{j-1}) \\
&\ge \sum_{j=1}^{k} \big(M\big(f, [x_j - x_{j-1}]\big) - m\big(f, [x_j - x_{j-1}]\big)\big)(x_j - x_{j-1}) \\
&= U(f, P_1) - L(f, P_1),
\end{aligned}$$

where the inequality comes from the fact that all the terms in the sum are positive. Hence, for any $\epsilon > 0$, we have found a partition P_1 such that

$$0 \le U(f, P_1) - L(f, P_1) < \epsilon.$$

This proves that f is integrable on $[a, c]$. The proof that f is integrable on $[c, b]$ is very similar and is omitted.

We now turn to the proof of S3.

Let P be a partition of $[a, b]$. Let Q be the partition P to which we add c. Let P_1 be the set of points in Q that are also in $[a, c]$. Let P_2 be the set of points in Q that are also in $[c, b]$. Then, P_1 and P_2 are partitions of $[a, c]$ and $[c, b]$, respectively. It is easy to check that

$$U(f, Q) = U(f, P_1) + U(f, P_2).$$

Recall that for an integrable function f and $x < y$, $\int_x^y f$ is the greatest lower bound of the set of upper Darboux sums on $[x, y]$. Since Q is a refinement of P, we have

$$U(f, P) \geq U(f, Q) = U(f, P_1) + U(f, P_2) \geq \int_a^c f + \int_c^b f.$$

Hence, $\int_a^c f + \int_c^b f$ is a lower bound of the set of upper Darboux sums on $[a, b]$. Since $\int_a^b f$ is the greatest lower bound of that set, we have

$$\int_a^b f \geq \int_a^c f + \int_c^b f.$$

Since f is integrable, so is $-f$. We apply the preceding inequality to $-f$ to get

$$\int_a^b (-f) \geq \int_a^c (-f) + \int_c^b (-f).$$

Multiplying by -1 both sides and using S1 yields

$$\int_a^b f \leq \int_a^c f + \int_c^b f.$$

This completes the proof of S3.

The integral is increasing

Assume that f and g are integrable on $[a, b]$. Suppose that for all x in $[a, b]$, we have

$$f(x) \leq g(x).$$

Then

$$\int_a^b f \leq \int_a^b g.$$

Let $h = g - f$. Then h is integrable on $[a, b]$ (why?), and $h(x) \geq 0$ for all x in $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Since h is positive, we have, for all $i = 1, \ldots, n$,

$$m\big(h, [x_{i-1}, x_i]\big) \geq 0.$$

Hence,

$$L(h, P) \geq 0.$$

Since $\int_a^b h$ is larger than $L(h, P)$, we have

$$\int_a^b h \geq 0.$$

We now use S1 and S2 to get

$$\int_a^b h = \int_a^b g - \int_a^b f \geq 0.$$

This completes the proof.

Application 6.1 Assume that f is continuous and positive on $[a, b]$. If $\int_a^b f = 0$, then f is identically 0 on $[a, b]$.

We prove the contrapositive. Assume that f is not identically 0. Since f is assumed to be positive, there must be a c in $[a, b]$ such that $f(c) > 0$. By Application 5.1 in Sect. 5.1, there is a $\delta > 0$ such that if $|x - c| < \delta$, then $f(x) > f(c)/2$. That is, f is bounded away from 0 near c. Suppose that c is in (a, b). If c is one of the end points, the proof is easily modified. By S3 we have

$$\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^b f.$$

Since f is positive, these three integrals are positive. Therefore,

$$\int_a^b f \geq \int_{c-\delta}^{c+\delta} f.$$

Using now that $f(x) > f(c)/2$ for all x in $[x - \delta, x + \delta]$, we have

$$\int_a^b f \geq \int_{c-\delta}^{c+\delta} f \geq \int_{c-\delta}^{c+\delta} f(c)/2 = \delta f(c) > 0.$$

Hence, $\int_a^b f > 0$, and Application 6.1 is proved.

What if $a > b$ or $a = b$?
Assume that f is integrable on $[b, a]$, then we define $\int_a^b f$ by

$$\int_a^b f = -\int_b^a f.$$

We also define

$$\int_a^a f = 0.$$

We now state another important property.

Composition and integrability
Assume that f is integrable on $[a, b]$ and that g is continuous on the range of f. Then $g \circ f$ is integrable on $[a, b]$.

The proof of this result is a little involved, and so we omit it (see, for instance, Theorem 6.11 in Principles of Mathematical Analysis, Third Edition McGraw-Hill, by W. Rudin). There is, however, an important particular case which is easy to prove: If we assume f to be continuous (which is more restrictive than integrable), then we know that $g \circ f$ is continuous and therefore integrable on $[a, b]$.

Absolute value and integrability
Suppose that f is integrable on $[a, b]$. Then $|f|$ is also integrable, and

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Let g be defined by $g(x) = |x|$. Then g is continuous everywhere, and $g \circ f = |f|$ is integrable on $[a, b]$, according to the preceding property. We now prove the inequality. We use the elementary fact:

$$|x| = \max(x, -x)$$

for every real x. In particular, we have $x \leq |x|$ and $-x \leq |x|$.

There are two cases to consider. Suppose first that

$$\int_a^b f \geq 0.$$

Then

$$\left|\int_a^b f\right| = \int_a^b f.$$

Note that $f \leq |f|$; hence,

$$\left|\int_a^b f\right| = \int_a^b f \leq \int_a^b |f|.$$

The other possibility is that

$$\int_a^b f \leq 0.$$

Then,

$$\left|\int_a^b f\right| = -\int_a^b f \leq \int_a^b |f|$$

since $-f \le |f|$. This completes the proof.

Integral Mean Value Theorem
Suppose that f is continuous on $[a, b]$. Then there is a c in $[a, b]$ such that

$$\int_a^b f = f(c)(b-a).$$

Since f is continuous on a closed and bounded interval $[a, b]$ the extreme value theorem applies. There are reals x_0 and x_1 in $[a, b]$ where f attains its minimum and its maximum, respectively. That is,

$$m\big(f, [a, b]\big) = f(x_0), \qquad M\big(f, [a, b]\big) = f(x_1).$$

We have

$$m\big(f, [a, b]\big) \le f(x) \le M\big(f, [a, b]\big) \quad \text{for all } x \in [a, b].$$

Taking integrals across these inequalities yields

$$m\big(f, [a, b]\big)(b-a) \le \int_a^b f \le M\big(f, [a, b]\big)(b-a).$$

Therefore,

$$m\big(f, [a, b]\big) = f(x_0) \le \frac{1}{b-a}\int_a^b f \le M\big(f, [a, b]\big) = f(x_1).$$

That is, the real

$$\frac{1}{b-a}\int_a^b f$$

is between $f(x_0)$ and $f(x_1)$. According to intermediate value theorem, the continuous function f assumes all values in $[f(x_0), f(x_1)]$. Therefore, there is a c between x_0 and x_1 such that

$$f(c) = \frac{1}{b-a}\int_a^b f.$$

This completes the proof of the integral mean value theorem.

Application 6.2 Assume that f is continuous on $[a, b]$. Let F be the function

$$F(x) = \int_a^x f.$$

Then F is differentiable on (a, b), and for all x in (a, b) we have

$$F'(x) = f(x).$$

Note that f is continuous on $[a, x]$ and therefore integrable on $[a, x]$ for all x in $[a, b]$. Hence, F is defined on $[a, b]$. Take c in (a, b) and let x_n be a sequence in $[a, b]$ that converges to c and such that $x_n \neq c$ for all n. We have, by the additivity property,

$$F(x_n) - F(c) = \int_a^{x_n} f - \int_a^c f = \int_a^c f + \int_c^{x_n} f - \int_a^c f = \int_c^{x_n} f.$$

Since f is continuous, we may apply the integral mean value theorem: there exists c_n between c and x_n such that

$$F(x_n) - F(c) = f(c_n)(x_n - c).$$

Since c_n is between c and x_n, we must have

$$|c_n - c| \le |x_n - c|.$$

Using that x_n converges to c, we get that c_n must converge to c. By the continuity of f at c, $f(c_n)$ converges to $f(c)$. Therefore,

$$\frac{F(x_n) - F(c)}{x_n - c} = f(c_n)$$

converges to $f(c)$. This proves that F is differentiable at c and that

$$F'(c) = f(c).$$

Example 6.6 Consider $f(x) = |x|$ on $[-1, 1]$. Then f is continuous on $[-1, 1]$ but not differentiable at 0. According to Application 6.2,

$$F(x) = \int_{-1}^{x} f$$

is differentiable on $(-1, 1)$, and

$$F'(x) = |x| \quad \text{for all } x \in (-1, 1).$$

In particular, F is differentiable at 0, and $F'(0) = 0$.

Exercises

1. We have shown that $m(cf) = cM(f)$ for f bounded and $c < 0$. Show that this implies $M(cf) = cm(f)$.
2. Prove part (iv) of Lemma 6.5.
3. In this exercise we prove S1 for $c < 0$. Assume that f is integrable on $[a, b]$.
   (a) Show for any partition P that $U(-f, P) = -L(f, P)$ and that $L(-f, P) = -U(f, P)$.
   (b) Show that $-f$ is integrable.
   (c) Show that for any partition P,

$$L(f, P) - U(f, P) \le \int_a^b (-f) + \int_a^b f \le U(f, P) - L(f, P).$$

   (d) Conclude that S1 holds for $c = -1$ and then for all $c < 0$.

4. (a) Give an example for which

$$M(f+g) < M(f) + M(g).$$

 (b) Give an example for which

$$M(f+g) = M(f) + M(g).$$

5. Assume that $a < c < b$ and that f is integrable on $[a,c]$ and on $[c,b]$. Prove that f is integrable on $[a,b]$.
6. Give an example of a function f which is positive and integrable on $[a,b]$, which is not identically 0, but whose integral is 0. Does this contradict Application 6.1?
7. Show that S3 holds even when $c < a$ or $c > b$.
8. Suppose that f is integrable on $[a,b]$. Show that

$$m\big(f,[a,b]\big)(b-a) \le \int_a^b f \le M\big(f,[a,b]\big)(b-a).$$

9. Give an example of function f for which

$$\left|\int_a^b f\right| < \int_a^b |f|.$$

10. Assume that f and g are integrable on $[a,b]$.
 (a) Show that

$$(f+g)^2 - (f-g)^2 = 4fg.$$

 (b) Show that fg is integrable.
11. Assume that f is integrable on $[a,b]$ and that $|f(x)| < K$ for all x in $[a,b]$. Show that for all x and y in $[a,b]$, we have

$$\left|\int_x^y f\right| \le K|x-y|.$$

12. Suppose that f and g are differentiable and that f' and g' are continuous on $[a,b]$. Prove the integration-by-parts formula

$$\int_a^b (fg') = \int_a^b (f'g) + f(b)g(b) - f(a)g(a).$$

13. Assume that f is continuous on $[a,b]$ and that for all x in $[a,b]$,

$$\int_a^x f = 0.$$

 Show that f is identically 0 on $[a,b]$.
14. Let f and g be continuous on $[a,b]$ and assume that $g \ge 0$ on $[a,b]$.
 (a) Show that there are x_0 and x_1 in $[a,b]$ such that for all x in $[a,b]$,

$$f(x_0) \le f(x) \le f(x_1).$$

(b) Assume that $\int_a^b g > 0$. Show that

$$f(x_0) \le \frac{\int_a^b (fg)}{\int_a^b g} \le f(x_1).$$

(c) Prove that there is c in $[a, b]$ such that

$$\int_a^b (fg) = f(c) \int_a^b g.$$

(d) Show that (c) holds when $\int_a^b g = 0$ as well. (Use Application 6.1.)

15. We define a distance between continuous functions f and g by setting

$$d(f, g) = \int_a^b |f - g|.$$

(a) Show that if f and g are continuous on $[a, b]$, then $|f - g|$ is continuous and therefore integrable.
(b) Show that $d(f, g) \ge 0$ for all continuous functions f and g.
(c) Show that for all continuous functions f and g, $d(f, g) = 0$ if and only if $f = g$ on $[a, b]$. (Use Application 6.1.)
(d) Show that

$$d(f, g) = d(g, f)$$

for all continuous functions f and g.
(e) Show that

$$d(f, g) \le d(f, h) + d(h, g)$$

for all continuous functions f, g, and h.

16. Assume that f and g are integrable on $[a, b]$.
(a) Show that

$$\int_a^b (f - g)^2 \ge 0.$$

(b) Use (a) to show that

$$\int_a^b (fg) \le \frac{1}{2}\left(\int_a^b f^2 + \int_a^b g^2\right).$$

6.3 Uniform Continuity

We used the notion of uniform continuity in Sect. 6.1 to prove the Riemann integrability of continuous functions. In this section we discuss this notion more in depth. We start with the definition.

Uniformly continuous functions
A function f is said to be uniformly continuous on the set D if for every $\epsilon > 0$, there is a $\delta > 0$ such that if x and y are in D and if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

As noted in Sect. 6.1, uniform continuity concerns the behavior of a function on a whole set. Continuity at a point a, on the other-hand, depends on the behavior of the function near a. It is clear that if f is uniformly continuous on the set D, then f is continuous at every point a of D. However, as we will show below, the converse is not true.

Example 6.7 The function f defined on $\mathbf{R}$ by $f(x) = x$ is uniformly continuous on $\mathbf{R}$.

This is very easy: for any $\epsilon > 0$, set $\delta = \epsilon$. If $|x - y| < \delta$, then

$$\left|f(x) - f(y)\right| = |x - y| < \delta = \epsilon,$$

and we are done.

We now state an useful criterion to prove that a function is NOT uniformly continuous.

Nonuniform continuity criterion
The function f is not uniformly continuous on the set D if and only if there are two sequences x_n and y_n in D such that $x_n - y_n$ converges to 0 and such that $f(x_n) - f(y_n)$ is bounded away from 0. That is, there is $a > 0$ such that for all $n \geq 1$,

$$\left|f(x_n) - f(y_n)\right| \geq a.$$

Assume first that f is not uniformly continuous on D. Then, there is an $\epsilon > 0$ such that for every $\delta > 0$, there are x and y in D such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$. For every $n \geq 1$, we may pick $\delta = 1/n$, and the corresponding x and y can be denoted by x_n and y_n. We construct like this two sequences x_n and y_n in D such that for all $n \geq 1$,

$$|x_n - y_n| < 1/n \quad \text{and} \quad \left|f(x_n) - f(y_n)\right| \geq \epsilon.$$

Hence, $x_n - y_n$ converges to 0, and $f(x_n) - f(y_n)$ is bounded away from 0. This proves the direct implication.

For the converse, we do a proof by contradiction. Assume that f is uniformly continuous and that there are two sequences x_n and y_n in D such that $x_n - y_n$ converges to 0 and that there is $a > 0$ such that for all $n \geq 1$,

$$\left|f(x_n) - f(y_n)\right| \geq a.$$

For $\epsilon = a > 0$, there must be a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < a$. Since $x_n - y_n$ converges to 0, there is a natural N such that if $n \geq N$, then

$$|x_n - y_n| < \delta.$$

Hence, for $n \geq N$,

$$\left|f(x_n) - f(y_n)\right| < a.$$

But we know that $|f(x_n) - f(y_n)| \geq a$. We have a contradiction, and thus f is not uniformly continuous on D. This completes the proof.

We apply the nonuniform continuity criterion in the next example.

Example 6.8 Let h be defined by $h(x) = 1/x$ for x in $(0, 1]$. Show that h is not uniformly continuous on $(0, 1]$.

By looking at the graph of h one can guess that this function is not uniformly continuous because of its behavior near 0. As we approach 0, the function grows faster and faster to infinity. This suggests that we pick sequences x_n and y_n that approach 0. Let $x_n = 1/n$ and $y_n = 2/n$. Therefore, $x_n - y_n = -1/n$ converges to 0. On the other hand,

$$\left|h(x_n) - h(y_n)\right| = n/2 \geq 1/2$$

for all $n \geq 1$. Hence, we have found two sequences x_n and y_n such that $x_n - y_n$ converges to 0 and $f(x_n) - f(y_n)$ bounded away from 0. By the nonuniform continuity criterion, h is not uniformly continuous on $(0, 1]$.

We now state the most important result of this section.

Continuity on a closed and bounded interval
Assume that f is continuous at every point of a closed and bounded interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.

We do a proof by contradiction. Assume that f is continuous but not uniformly continuous on $[a, b]$. By the nonuniform continuity criterion, there are two sequences x_n and y_n on $[a, b]$ such that $x_n - y_n$ converges to 0 and such that there is an $\epsilon > 0$ for which

$$\left|f(x_n) - f(y_n)\right| \geq \epsilon \quad \text{for all } n \geq 1.$$

Since x_n is a bounded sequence (why?), by the Bolzano–Weierstrass theorem, it has a convergent subsequence x_{n_k}. The limit ℓ must be in $[a, b]$ (why?). On the other hand, $x_{n_k} - y_{n_k}$ is a subsequence of $x_n - y_n$ and therefore must converge to 0. Hence,

$$y_{n_k} = y_{n_k} - x_{n_k} + x_{n_k} \quad \text{converges to } 0 + \ell = \ell.$$

By the continuity of f at ℓ we have that

$$f(x_{n_k}) \quad \text{and} \quad f(y_{n_k}) \quad \text{both converge to } \ell.$$

Hence, $f(x_{n_k}) - f(y_{n_k})$ converges to 0. Therefore, there is a natural K such that if $k \geq K$, then

$$\left|f(x_{n_k}) - f(y_{n_k})\right| < \epsilon/2.$$

This contradicts $|f(x_n) - f(y_n)| \geq \epsilon$ for every n. The function f must be uniformly continuous on $[a, b]$.

Note that Example 6.8 shows that it is crucial for the set to be closed for the theorem to hold. The function $1/x$ is continuous on (0, 1] but not uniformly continuous. It is also crucial for the set to be bounded. In the Exercises, one shows that the function x^2 is continuous on $\mathbf{R}$ but not uniformly continuous.

Lipschitz functions
Assume that there is $K > 0$ such that

$$\left|f(x) - f(y)\right| \leq K|x - y|$$

for all x and y in D. Then, f is said to be a Lipschitz function. If f is a Lipschitz function on D, then it is uniformly continuous.

We now prove that if f is Lipschitz on D, then it is uniformly continuous. Let $\epsilon > 0$ and pick $\delta = \epsilon/K$. If $|x - y| < \delta$, then

$$\left|f(x) - f(y)\right| \leq K|x - y| < K\delta = \epsilon.$$

Hence, the function f is uniformly continuous, and we are done.

We apply the preceding result in the next example.

Example 6.9 Show that the function f defined $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$.

Observe that

$$\left|f(x) - f(y)\right| = \left|\sqrt{x} - \sqrt{y}\right| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}}.$$

Since $x \geq 1$, we have $\sqrt{x} \geq \sqrt{1} = 1$. Therefore,

$$\sqrt{x} + \sqrt{y} \geq 2,$$

and since the inverse function is decreasing on the positive reals, we have

$$\frac{1}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2}.$$

Hence,

$$\left|f(x) - f(y)\right| \leq \frac{1}{2}|x - y|.$$

Therefore, f is Lipschitz (with $K = 1/2$) and so uniformly continuous on $[1, \infty)$.

We will show in the exercises that a function may be uniformly continuous without being Lipschitz.

Exercises

1. Let g be defined by $g(x) = x^2$.
   (a) Show that g is not uniformly continuous on **R**.
   (b) Is there a set on which g is uniformly continuous?
2. Show that the function h defined by $h(x) = 1/x$ is uniformly continuous on $[1, \infty)$.
3. Let the function f be defined by $f(x) = \sqrt{x}$ on [0, 1].
   (a) Show that there is a sequence x_n in (0, 1] such that $f(x_n)/x_n$ is unbounded.
   (b) Use (a) to show that the function f is not Lipschitz on [0, 1].
   (c) Show that f provides an example of a function which is not Lipschitz on [0, 1] but which is uniformly continuous.
4. Assume that f and g are uniformly continuous on D.
   (a) Show that $f + g$ is uniformly continuous on D.
   (b) Is fg necessarily uniformly continuous on D?
5. Assume that f is uniformly continuous on [0, 1] and on $[1, \infty)$. Show that f is uniformly continuous on $[0, \infty)$.
6. Show that the function f defined by $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$. (Use Exercise 5.)
7. Let f be Riemann integrable on [0, 1]. Define
$$F(x) = \int_0^x f \quad \text{for } x \in [0, 1].$$
   Show that F is a Lipschitz function. (Recall that f is bounded.)
8. Let $g(x) = \sin(1/x)$ for x in (0, 1).
   (a) Show that g is continuous on (0, 1).
   (b) Prove that g is not uniformly continuous on (0, 1).
9. Let f be uniformly continuous on $[0, \infty)$. In this exercise we will show that f can grow at most linearly.
   (a) Show that there is a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < 1$. Let $d = \delta/2$.
   (b) Show that
$$\big|f(d)\big| < 1 + \big|f(0)\big|.$$
   (c) Prove that for every natural n, we have
$$\big|f(nd)\big| < n + \big|f(0)\big|.$$
   (d) For any x in $[0, \infty)$, there is a positive integer $n(x)$ such that
$$n(x) \le x/d < n(x) + 1.$$
   (For a proof, see the lemma in Sect. 1.3.) Show that
$$\big|f(x)\big| < 1 + \big|f\big(n(x)d\big)\big| < 1 + n(x) + \big|f(0)\big| < x/d + \big|f(0)\big| + 1.$$
   The function on the r.h.s. is linear in x, and this concludes the proof.

Chapter 7
Convergence of Functions

In many situations we have a sequence of functions f_n that converges to some function f and f is not easy to study directly. Can we use the functions f_n to get some information about f? For instance, if the f_n are continuous, is f necessarily continuous? Another question that often comes up is: can I compute $\int_a^b f\,dx$ using $\int_a^b f_n\,dx$? More precisely, is it true that

$$\lim_{n\to\infty}\int_a^b f_n\,dx = \int_a^b f\,dx?$$

We can rewrite the question as

$$\lim_{n\to\infty}\int_a^b f_n\,dx = \int_a^b \lim_{n\to\infty} f_n\,dx?$$

In other words, can we interchange the limit and the integral? We will give some partial answers to these questions in this chapter. First, we need to define what we mean by the convergence of a sequence of functions. There are many different ways a sequence of functions can converge. In this chapter we will just consider two of them. Here is the first.

Pointwise convergence
Consider a sequence of functions f_n all defined on the same set $S \subset \mathbf{R}$. The sequence f_n is said to converge pointwise to f on S if for every x, in S we have that

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Observe that we are using the definition of numerical sequences $f_n(x)$ to define the convergence of a sequence of functions f_n. More precisely, if f_n converges pointwise to f on S, it means that for every (fixed) x in S and every $\epsilon > 0$, there exists N (that depends on ϵ and x) such that for $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$.

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Example 7.1 Consider the sequence $f_n(x) = e^{-nx}$ and take $S = [0, \infty)$. Does the sequence f_n converge?

We have

$$f_n(x) = e^{-nx} = \left(e^{-x}\right)^n = r^n,$$

where $r = e^{-x}$. There are two cases. If $x = 0$, then $r = 1$ and $r^n = 1$. Therefore, $f_n(0)$ converges to 1. On the other hand, if $x > 0$, then $r < 1$, and r^n converges to 0. That is, $f_n(x)$ converges to 0 for all $x > 0$. In summary, we have proved that f_n converges pointwise to the function f on $[0, \infty)$ defined by $f(0) = 1$ and $f(x) = 0$ for $x > 0$.

Example 7.1 shows that a sequence of continuous functions f_n may converge pointwise to a limit f which is not continuous!

Example 7.2 Consider the sequence f_n defined on $[0, 1]$ by

$$\begin{aligned} &f_n(0) = 0, \\ &f_n(x) = n \quad \text{for } x \in \left(0, \frac{1}{n}\right], \\ &f_n(x) = 0 \quad \text{for } x \in \left(\frac{1}{n}, 1\right]. \end{aligned}$$

We first show that f_n converges pointwise to 0 on $[0, 1]$. If $x = 0$, then $f_n(0)$ is always 0 and therefore converges to 0. If x is in $(0, 1]$, then there exists N such that $N > \frac{1}{x}$ (why?) and therefore $x > \frac{1}{N}$. Hence, for all $n \geq N$, we have $x > \frac{1}{n}$. By the definition of f_n we have $f_n(x) = 0$ for all $n \geq N$. This implies that $f_n(x)$ converges to 0. We have proved that f_n converges pointwise to 0 on $[0, 1]$.

We now turn to a different question. Is it true that

$$\lim_{n\to\infty} \int_0^1 f_n \, dx = \int_0^1 \lim_{n\to\infty} f_n \, dx?$$

We will show that this is not true for this example. First note that each f_n is Riemann integrable on $[0, 1]$ and that

$$\int_0^1 f_n(x) \, dx = \int_0^{1/n} n \, dx = 1.$$

Hence, the sequence $\int_0^1 f_n(x) \, dx$ is always 1 and therefore converges to 1. On the other hand, we know that for every x in $[0, 1]$, $f_n(x)$ converges to 0, so

$$\int_0^1 \lim_{n\to\infty} f_n \, dx = \int_0^1 0 \, dx = 0.$$

This is an example for which we cannot interchange the limit and the integral.

Examples 7.1 and 7.2 show that pointwise convergence is not enough for our purposes. A sequence of continuous functions need not converge to a continuous function, and the interchange of limit and integral need not be true. This is why we turn to a more stringent type of convergence.

Uniform convergence
Consider a sequence of functions f_n all defined on the same set $S \subset \mathbf{R}$. The sequence f_n is said to converge uniformly to f on S if for every $\epsilon > 0$, there exists a natural N such that for all $n \geq N$, we have

$$\left|f_n(x) - f(x)\right| < \epsilon \quad \text{for all } x \in S.$$

The critical difference between pointwise and uniform convergence is the following. In the case of uniform convergence we ask for the *same* N for all x in S. This is why it is called *uniform* convergence. In the case of pointwise convergence, for each x in S, we have a possibly different N. The following criterion will be useful in checking for uniform convergence.

Uniform convergence criterion
Consider a sequence of functions f_n. For each $n \geq 1$, f_n is defined on S. For a fixed function f defined on S and for $n \geq 1$, let

$$m_n = \sup\left\{\left|f_n(x) - f(x)\right| : x \in S\right\}.$$

The sequence f_n converges uniformly to f if and only if m_n converges to 0.

We now prove the criterion. Assume first that f_n converges uniformly to f on S. Let $\epsilon > 0$. There exists a natural N such that for all $n \geq N$, we have

$$\left|f_n(x) - f(x)\right| < \epsilon \quad \text{for all } x \in S.$$

This shows that for $n \geq N$, the set $A_n = \{|f_n(x) - f(x)| : x \in S\}$ is bounded above by ϵ. Hence the supremum (i.e., the least upper bound) m_n of this set exists (why?) and is less than the upper bound ϵ. That is, for $n \geq N$, we have $m_n = |m_n - 0| \leq \epsilon$. This proves that m_n converges to 0. The direct implication is proved.

We now prove the converse. Assume that m_n converges to 0. Let $\epsilon > 0$. There is a natural N such that if $n \geq N$, then $|m_n - 0| < \epsilon$. Since m_n is positive, we have $m_n < \epsilon$. That is, the least upper bound m_n of the set A_n is less than ϵ for $n \geq N$. Hence, ϵ is an upper bound for A_n. Therefore, for $n \geq N$, we have

$$\left|f_n(x) - f(x)\right| \leq \epsilon$$

for all x in S. This proves that f_n converges uniformly to f on S. The proof of the criterion is complete.

It is easy to see that uniform convergence implies pointwise convergence. We will see shortly that the converse is not true.

Note that the uniform criterion does not tell us how to find f. Usually, we will proceed in two steps. First, we will compute the pointwise limit f of f_n. Then we will use the criterion to decide whether the convergence to f is uniform. Next, we give an example of the method.

Example 7.3 Consider the sequence of functions f_n defined for all reals by

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

Does f_n converge uniformly on $\mathbf{R}$?

We first compute the pointwise limit (if any). Let x be a fixed real number. As n goes to infinity, $x^2 + \frac{1}{n}$ converges to x^2, which is a positive number. Since the square root function is continuous on $[0, \infty)$, we get that $f_n(x)$ converges to $\sqrt{x^2} = |x|$. That is, the sequence f_n converges pointwise on $\mathbf{R}$ to the function f defined by $f(x) = |x|$.

Now that we have a function f, the second step is to estimate m_n. We compute

$$\left|f_n(x) - f(x)\right| = \left|\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}\right| = \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}}.$$

Since $x^2 \geq 0$, we have, for all x in $\mathbf{R}$,

$$\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2} \geq \sqrt{\frac{1}{n}},$$

and the minimum $\sqrt{\frac{1}{n}}$ is attained at $x = 0$. Hence, for all x in $\mathbf{R}$,

$$\left|f_n(x) - f(x)\right| \leq \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{n}},$$

and the maximum is attained at $x = 0$. This shows that $\frac{1}{\sqrt{n}}$ is the maximum (and therefore the least upper bound) of the set $\{|f_n(x) - f(x)| : x \in \mathbf{R}\}$. Hence, $m_n = \frac{1}{\sqrt{n}}$ for $n \geq 1$. The sequence m_n converges to 0, and by the uniform convergence criterion, f_n converges uniformly to f.

We now show that unlike pointwise convergence uniform convergence preserves continuity.

Uniform convergence preserves continuity
Assume that the sequence f_n converges uniformly to f on S. Assume that for $n \geq 1$, the function f_n us continuous at $a \in S$. Then f is also continuous at a.

We now prove this theorem. Let $\epsilon > 0$. By the uniform convergence criterion, the sequence m_n converges to 0. Hence, there is a natural N such that if $n \geq N$, then

$$|m_n - 0| = m_n < \epsilon/3.$$

Let $n_1 = N + 1$ (n_1 could be any natural larger than N). Since f_{n_1} is continuous at a, there is exists a $\delta > 0$ such that if $x \in S$ and $|x - a| < \delta$, then

$$\left|f_{n_1}(x) - f_{n_1}(a)\right| < \epsilon/3.$$

We add and subtract terms to get

$$\left|f(x) - f(a)\right| = \left|f(x) - f_{n_1}(x) + f_{n_1}(x) - f_{n_1}(a) + f_{n_1}(a) - f(a)\right|.$$

We now use the triangle inequality:

$$\left|f(x) - f(a)\right| \le \left|f(x) - f_{n_1}(x)\right| + \left|f_{n_1}(x) - f_{n_1}(a)\right| + \left|f_{n_1}(a) - f(a)\right|.$$

Recall now that m_n is the least upper bound of the set $\{|f_n(x) - f(x)| : x \in S\}$; therefore, $|f(x) - f_{n_1}(x)|$ and $|f_{n_1}(a) - f(a)|$ are both less than m_{n_1}. Hence,

$$\left|f(x) - f(a)\right| \le m_{n_1} + \left|f_{n_1}(x) - f_{n_1}(a)\right| + m_{n_1}.$$

Since $n_1 \ge N$, we know that $m_{n_1} < \epsilon/3$. And by continuity of f_{n_1} we have that $|f_{n_1}(x) - f_{n_1}(a)| < \epsilon/3$ for $|x - a| < \delta$. That is,

$$\left|f(x) - f(a)\right| \le \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

for x in S such that $|x - a| < \delta$. This proves that f is continuous at a.

Example 7.4 We revisit Example 7.1. Consider the sequence $f_n(x) = e^{-nx}$ on $S = [0, \infty)$. We showed that f_n converges to f pointwise, where f is defined by $f(0) = 1$ and $f(x) = 0$ for $x > 0$. Is the convergence uniform?

Each f_n is clearly continuous on S, however f is not. Hence, the convergence cannot be uniform. If the convergence were uniform, continuity would be preserved according to the theorem we just proved. This example shows that pointwise convergence does not imply uniform convergence.

Uniform convergence and integration

Assume that for each $n \ge 1$, the function f_n is continuous on the closed bounded interval $[a, b]$. Assume also that the sequence f_n converges uniformly to f on $[a, b]$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x) = \int_a^b f(x)\,dx.$$

That is, we can interchange limit and integral when we have uniform convergence. We now prove this.

Given that the functions f_n are assumed to be continuous, they are Riemann integrable. Since uniform convergence preserves continuity, f is continuous and therefore integrable as well.

Let $\epsilon > 0$. By the uniform convergence criterion, there is a natural N such that if $n \ge N$, then $|m_n - 0| = m_n < \epsilon/(b - a)$. Now consider

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| = \left|\int_a^b \big(f_n(x) - f(x)\big)\,dx\right| \le \int_a^b \left|f_n(x) - f(x)\right|dx,$$

where the last inequality is the well-known inequality

$$\left|\int_a^b g(x)\,dx\right| \le \int_a^b \left|g(x)\right|dx$$

valid for any Riemann-integrable function g. By the definition of m_n, we have, for every x in $[a,b]$,

$$\left|f_n(x) - f(x)\right| \le m_n,$$

and therefore,

$$\int_a^b \left|f_n(x) - f(x)\right|dx \le \int_a^b m_n\,dx = m_n(b-a).$$

For $n \ge N$, we have $m_n < \epsilon/(b-a)$, and therefore,

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \le \int_a^b \left|f_n(x) - f(x)\right|dx \le m_n(b-a) < \epsilon.$$

This proves that $\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$, and we are done.

We will apply this theorem to power series. We now turn to differentiability and uniform convergence. First, an example.

Example 7.5 We revisit Example 7.3. Consider the sequence of functions f_n defined for all reals by

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

We first show that for each $n \ge 1$, the function f_n is differentiable at $x = 0$ (in fact, a very similar argument shows that f_n is differentiable everywhere). Let g_n be defined by $g_n(x) = x^2 + \frac{1}{n}$. The function g_n is a polynomial and is therefore differentiable everywhere and hence at $x = 0$. Observe that $g_n(0) = \frac{1}{n}$ and that the function square root is differentiable at $\frac{1}{n}$. In fact the square root function is differentiable at any strictly positive number (but not at 0). Hence, by the chain rule, $f_n = \sqrt{g_n}$ is differentiable at $x = 0$.

Observe now that in Example 7.3 we have proved that f_n converges uniformly to the absolute value function, which is not differentiable at 0. This example shows that uniform convergence does not preserve differentiability!

The next result gives sufficient conditions for the limit function to be differentiable. First, a definition. A function g is said to be C^1 on (a,b) if the function g is differentiable on (a,b) and the function g' is continuous on (a,b).

Uniform convergence and differentiation
Assume that for $n \ge 1$, the function f_n is C^1 on (a,b). Assume that the sequence f_n converges pointwise on (a,b) to a function f. Finally, assume that the sequence of derivatives f_n' converges uniformly on (a,b) to some function g. Then the function f is C^1, and $f' = g$.

Note that we do not require f_n to converge uniformly. We now prove this theorem. By the fundamental theorem of Calculus,

$$\int_c^x f_n'(t)\,dt = f_n(x) - f_n(c) \tag{7.1}$$

for any x and c in (a,b). Since f_n' converges uniformly to g on (a,b) and therefore on $[c,x]$ (why?), the result on uniform convergence and integration gives

$$\lim_{n\to\infty} \int_c^x f_n'(t)\,dt = \int_c^x g(t)\,dt.$$

On the other hand, since f_n converges pointwise to f on (a,b), we have

$$\lim_{n\to\infty} \big(f_n(x) - f_n(c)\big) = f(x) - f(c).$$

Using the two limits above and letting n go to infinity in (7.1), we get

$$\int_c^x g(t)\,dt = f(x) - f(c). \tag{7.2}$$

Observe now that the functions f_n' are assumed to be continuous and the sequence f_n' converges uniformly to g. Hence, g is continuous. This implies by another version of the fundamental theorem of Calculus that $\int_c^x g(t)\,dt$ is differentiable as a function of x and that its derivative is $g(x)$, see Application 6.2 in Sect. 6.2. By (7.2), $f(x) - f(c)$ is equal to a differentiable function. Hence, $f(x) - f(c)$ is also differentiable, and f is differentiable. By taking derivatives with respect to x on both sides of (7.2) we get

$$g(x) = f'(x).$$

Since g is continuous, so is f', and therefore f is C^1 on (a,b). This completes the proof of the theorem.

We will now apply the results above to power series.

Power Series We first recall a few facts from Sect. 3.1. A power series is a function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where c_n is a sequence of real numbers. The function f is defined at x if and only if the series converges at x. Note that $f(0) = c_0$, so the function is always defined at $x = 0$. For each power series, there is a so-called *radius of convergence* R. The power series converges if $|x| < R$, diverges if $|x| > R$, and can go either way if $|x| = R$. We may have $R = +\infty$, in which case the power series converges for all x. Finally, R is the least upper bound (if it exists!) of the set

$$I = \big\{r \geq 0 : \text{ the sequence } c_n r^n \text{ is bounded}\big\}.$$

In particular, if $|x| > R$, then the sequence $c_n x^n$ is not bounded.

For $n \geq 1$, we define

$$f_n(x) = \sum_{k=0}^{n} c_k x^k.$$

When we say that the power series converges at x, we mean that

$$\lim_{n \to \infty} f_n(x) = f(x).$$

Hence, this is pointwise convergence. In fact, as we are going to see now, we also have uniform convergence inside $(-R, R)$.

Uniform convergence for power series
Consider a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with a strictly positive radius of convergence R. For $n \geq 1$, let

$$f_n(x) = \sum_{k=0}^{n} c_k x^k$$

be the sequence of partial sums. Then, for any $0 < r < R$, the sequence f_n converges uniformly to f on $[-r, r]$.

We now prove this theorem. Consider

$$\left| f_n(x) - f(x) \right| = \left| \sum_{k=n+1}^{\infty} c_k x^k \right| \leq \sum_{k=n+1}^{\infty} \left| c_k x^k \right|.$$

Since $r < R$, we can find r_1 such that $r < r_1 < R$. Assume that $|x| \leq r < r_1$. Then

$$\left| c_k x^k \right| = |c_k||x|^k = |c_k| r_1^k \frac{|x|^k}{r_1^k} \leq |c_k| r_1^k \frac{r^k}{r_1^k}.$$

Since $r_1 < R$, we know that the sequence $|c_k| r_1^k$ is bounded (why?) by some M. Therefore,

$$\left| c_k x^k \right| \leq M \left(\frac{r}{r_1} \right)^k.$$

Hence, for all x in $[-r, r]$,

$$\left| f_n(x) - f(x) \right| \leq M \sum_{k=n+1}^{\infty} \left(\frac{r}{r_1} \right)^k = M \left(\frac{r}{r_1} \right)^{n+1} \frac{1}{1 - \frac{r}{r_1}}.$$

Let m_n be the least upper bound of the set $\{|f_n(x) - f(x)| : x \in [-r, r]\}$. We have

$$0 \leq m_n \leq M \left(\frac{r}{r_1} \right)^{n+1} \frac{1}{1 - \frac{r}{r_1}}.$$

By the squeezing principle, m_n converges to 0 as n goes to infinity (why?), and by the uniform convergence criterion, f_n converges uniformly to f on $[-r, r]$.

Power series and integration
Consider a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with a strictly positive radius of convergence R. Let $a < b$ be such that $[a, b] \subset (-R, R)$. Then f is integrable on $[a, b]$, and

$$\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} c_n \int_a^b x^n\,dx.$$

We now prove this result. First note that there is r in $(0, R)$ such that $[a, b] \subset [-r, r]$. Hence, the sequence of partial sums f_n converges uniformly on $[-r, r]$ and therefore on $[a, b]$. Observe also each f_n is a polynomial and is therefore continuous (and infinitely differentiable). Hence, the result on uniform convergence and integration applies, and we have

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x) = \int_a^b f(x)\,dx. \tag{7.3}$$

By the definition of f_n we have

$$\int_a^b f_n(x)\,dx = \int_a^b \left(\sum_{k=0}^{n} c_k x^k\right) dx = \sum_{k=0}^{n} c_k \int_a^b x^k\,dx.$$

The critical point here is that we can always interchange integral and sum if the sum has *finitely* many terms (provided that the functions are integrable, of course). To do so, we just use the linearity property of the integral. However, to do the interchange for a sum with infinitely terms, we need more. Here we use uniform convergence.

Back to the proof, the last equality shows that $\int_a^b f_n(x)\,dx$ is the partial sum of the series with general term $c_k \int_a^b x^k\,dx$. By (7.3) we know that this partial sum converges and that

$$\lim_{n\to\infty} \sum_{k=0}^{n} c_k \int_a^b x^k\,dx = \sum_{k=0}^{\infty} c_k \int_a^b x^k\,dx = \int_a^b f(x)\,dx.$$

This completes the proof.

An important application of the preceding result is to get new power series from known ones. The next example illustrates this.

Example 7.6 Recall from Application 4.7 in Sect. 4.1 that the binomial series yields

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} x^{2k}.$$

Moreover, this power series has a radius of convergence equal to 1. Let y be in $(-1, 1)$. Then $[0, y]$ is a subset of $(-1, 1)$, and by the result on power series and integration we can integrate the series term by term:

$$\int_0^y \frac{1}{\sqrt{1-x^2}}\,dy = \int_0^y 1\,dy + \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \int_0^y x^{2k}\,dy.$$

Now, an antiderivative of $\frac{1}{\sqrt{1-x^2}}$ is $\arcsin x$. Hence, by the fundamental theorem of Calculus,

$$\arcsin y - \arcsin 0 = y + \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \frac{y^{2k+1}}{2k+1}.$$

Since $\arcsin 0 = 0$, we have

$$\arcsin y = y + \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \frac{y^{2k+1}}{2k+1}$$

for all y in $(-1, 1)$. This power series was used in Sect. 4.1 to approximate π.

Power series and differentiation
Consider a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with a strictly positive radius of convergence R. Then f is differentiable on $(-R, R)$, and for x in $(-R, R)$, we have

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

We have already used this result in Chap. 3 when we computed the derivatives of the functions sine, cosine, and exponential, and in Chap. 4 when we computed the binomial series.

We now prove this. For $n \geq 1$, let g_n be the sequence of partial sums

$$g_n(x) = \sum_{k=1}^{n} k c_k x^{k-1}.$$

Our first step is to prove that g_n converges uniformly to

$$g(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

on $[-r, r]$ if $0 < r < R$. Let $r < r_1 < R$, we have

$$\left| k c_k x^{k-1} \right| = k|c_k| r_1^{k-1} \left(\frac{|x|}{r_1} \right)^{k-1}.$$

The sequence $|c_k|r_1^{k-1}$ is bounded (why?) by some M. Hence,

$$\left|kc_k x^{k-1}\right| \le Mk\left(\frac{|x|}{r_1}\right)^{k-1} < Mk\left(\frac{r}{r_1}\right)^{k-1}$$

for $|x| < r$. Consider now

$$\left|g(x) - g_n(x)\right| = \left|\sum_{k=n+1}^{\infty} kc_k x^{k-1}\right| \le \sum_{k=n+1}^{\infty} \left|kc_k x^{k-1}\right| \le M \sum_{k=n+1}^{\infty} k\left(\frac{r}{r_1}\right)^{k-1}$$

for $|x| < r$. Let m_n be the least upper bound of the set $\{|g_n(x) - g(x); x \in [-r, r]\}$. We have

$$0 \le m_n \le M \sum_{k=n+1}^{\infty} k\left(\frac{r}{r_1}\right)^{k-1}.$$

Our last step is to show that $\sum_{k=n+1}^{\infty} k(\frac{r}{r_1})^{k-1}$ converges to 0 as n goes to infinity. That is, we want to show that the series $\sum_{k=1}^{\infty} k(\frac{r}{r_1})^{k-1}$ is convergent. Since $r/r_1 < 1$, this is a simple consequence of the ratio test. By using the squeezing principle on the double inequality above we get that m_n converges to 0. This completes the proof that the sequence g_n converges uniformly to g on $[-r, r]$.

We now check that the two other hypotheses of the result on uniform convergence and differentiation hold. First, the sequence of partial sums

$$f_n(x) = \sum_{k=1}^{n} c_k x^k$$

is C^1 on $[-r, r]$ (the f_n are polynomials!). Second, the sequence f_n converges uniformly (and therefore pointwise) to f on $[-r, r]$. These facts, together with the uniform convergence on $[-r, r]$ of $f_n' = g_n$ that we just proved, imply that f is differentiable and $f' = g$ on $[-r, r]$ for any $r < R$.

Exercises

1. Assume that the numerical sequence a_n has the following property. There exists N such that $a_n = 0$ for all $n \ge N$. Show that a_n converges to 0.
2. Consider the sequence f_n defined on [0, 1]. Let $f_n(0) = 0$ and $f_n(x) = 0$ for $x \in (\frac{1}{n}, 1]$. We have no information on f_n on $(0, \frac{1}{n}]$. Prove that f_n converges pointwise to 0 on [0, 1].
3. Prove that uniform convergence implies pointwise convergence.
4. Let
$$f_n(x) = \frac{\sin(nx)}{n}.$$
Show that f_n converges uniformly on **R**.
5. Let
$$f_n(x) = xe^{-nx}.$$

(a) Show that f_n converges pointwise to 0 on $[0, \infty)$.
(b) Let $m_n = \sup\{|f_n(x)|; x \in [0, \infty)\}$. Show that $m_n = \frac{1}{ne}$.
(c) Conclude that f_n converges uniformly on $[0, \infty)$.

6. Consider the sequence $f_n(x) = e^{-nx}$ on $S = [1, \infty)$.
   (a) Show that f_n converges uniformly on S.
   (b) Show that for any $a > 0$, f_n converges uniformly on $[a, \infty)$.
   (c) Does f_n converge uniformly on $(0, \infty)$?
7. Consider the sequence of functions f_n on [0, 1] defined by $f_n(x) = x^n$.
   (a) Show that f_n converges pointwise to a function f on [0, 1].
   (b) Show that f_n does not converge uniformly on [0, 1].
   (c) Find a subset of [0, 1] on which f_n does converge uniformly.
8. Assume that the functions f_n are continuous on $[a, b]$ and that the sequence f_n converges uniformly to f on $[a, b]$. Let
$$F_n(x) = \int_a^x f_n(t)\,dt \quad \text{and} \quad F(x) = \int_a^x f(t)\,dt.$$
   (a) Show that the functions F_n and F are defined on $[a, b]$.
   (b) Prove that F_n converges uniformly to F on $[a, b]$.
9. Show that if f_n converges uniformly to f on S and $T \subset S$, then f_n converges uniformly to f on T as well.
10. Check that the hypotheses of the fundamental theorem of Calculus hold in (7.1).
11. Consider a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with a strictly positive radius of convergence R.
   (a) Show that for y in $(0, R)$, we have
$$\int_0^y f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} y^{n+1}.$$
   (b) Show that if $|y| < R$ then the series $\sum_{n=0}^{\infty} \frac{c_n}{n+1} y^{n+1}$ converges. (Use (a).)
   (c) Show that if $|y| > R$, then the sequence $\frac{c_n}{n+1} y^{n+1}$ is not bounded. (Take r in $(R, |y|)$ and show that the sequence $r^{n+1} \frac{c_n}{n+1} \frac{y^{n+1}}{r^{n+1}}$ is not bounded.)
   (d) Conclude that the series $\sum_{n=0}^{\infty} \frac{c_n}{n+1} y^{n+1}$ has radius of convergence R.
12. Show that uniform convergence preserves uniform continuity.
13. Consider a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with a strictly positive radius of convergence R. Show that the power series $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ has the same radius of convergence R. (Use the methods of Exercise 11.)
14. Consider a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with a strictly positive radius of convergence R.
   (a) Show that f is twice differentiable on $(-R, R)$ and that
$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

(b) Show that in fact for any $k \geq 1$, the function f can be differentiated k times, and its kth derivative $f^{(k)}$ is given by

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-k}.$$

(Do a proof by induction.)

Chapter 8
Decimal Representation of Numbers

So far we have freely used the decimal representation of numbers without even mentioning it. However, the decimal representation of real numbers has deep and interesting consequences. In particular, it shows that any real number can be represented by an infinite series. Some of the proofs in this chapter are long and a little technical. The reader should concentrate on the results and the examples.

We start by dealing with natural numbers.

Representation of naturals
For any natural number $n \geq 1$, there exists an integer $s \geq 0$ and integers $a_0, a_1, \dots, a_s$ in $\{0, 1, 2, \dots, 9\}$ with $a_s > 0$ such that

$$n = a_0 + a_1 10 + a_2 10^2 + \cdots + a_s 10^s.$$

$a_s a_{s-1} \dots a_1 a_0$ is the decimal representation of n. It is unique.

When we say that the number n *is* 121, we really mean that the decimal representation of n is 121. That is,

$$n = 1 + 2 \times 10 + 1 \times 10^2.$$

The usual convention is to identify numbers with their decimal representation. However numbers may be represented in different bases. For instance, in base 2, n is 1111001. That is,

$$n = 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2 + 1.$$

We now prove that every natural number has an unique decimal representation. We will need two steps: existence and uniqueness.

Existence Fix a natural number n. If $n \leq 9$, then we set $s = 0$, $a_0 = n$, and we have

$$n = a_0.$$

This is the decimal representation of n.

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Assume now that $n \geq 10$. Let

$$A = \left\{k \in \mathbf{N} : 10^k > n\right\}.$$

Since $10^n > n$ for all naturals, A is a nonempty set of naturals (n belongs to A). Therefore, it has a minimum element t such that $10^t > n$. Note that $t > 1$ since we are assuming that $n \geq 10$. Observe that $t - 1$ cannot be in A (otherwise t would not the minimum). There are two ways for $t - 1$ not to be in A: either it is not in $\mathbf{N}$, or $10^{t-1} \leq n$. Since t is in $\mathbf{N}$ and is at least 2, $t - 1$ is also in $\mathbf{N}$. Thus, we must have $10^{t-1} \leq n$. Therefore,

$$10^{t-1} \leq n < 10^t.$$

Set $s = t - 1$. We now need to perform Euclidean divisions. We first recall the properties.

Long division in the naturals
Given two natural numbers a and b, there exist positive integers q and r such that

$$a = bq + r$$

where $0 \leq r < b$. Moreover, q and r are unique. If $r = 0$, a is said to be divisible by b.

This was proved in Sect. 1.1.

We perform the Euclidean division of n by 10^s. There are unique positive integers a_s and r_s such that

$$n = a_s 10^s + r_s \quad \text{where } 0 \leq r_s < 10^s.$$

Note that

$$10^t = 10^{s+1} > n \geq a_s 10^s,$$

and so $a_s < 10$. Observe also that if $a_s = 0$, then $n = r_s < 10^s$, and this is a contradiction since $n \geq 10^s$. Thus, a_s is in $\{1, 2, \ldots, 9\}$. We now perform the Euclidean division of r_s by 10^{s-1}. There are positive integers a_{s-1} and r_{s-1} such that

$$r_s = a_{s-1} 10^{s-1} + r_{s-1} \quad \text{where } 0 \leq r_{s-1} < 10^{s-1}.$$

Since $r_s < 10^s$, we must have

$$a_{s-1} 10^{s-1} \leq r_s < 10^s.$$

Hence, $a_{s-1} < 10$. Note that

$$n = a_s 10^s + r_s = a_s 10^s + a_{s-1} 10^{s-1} + r_{s-1}.$$

We then divide r_{s-1} by 10^{s-2}, and so on. We denote the quotients of the successive divisions by $a_s, a_{s-1}, \ldots, a_1$. The last division we perform is r_2 by 10. The last remainder r_1 is therefore strictly less than 10, and we set $a_0 = r_1$. Using the successive divisions, we get

$$n = a_s 10^s + a_{s-1} 10^{s-1} + \cdots + a_1 10 + a_0,$$

where each a_i (for $1 \leq i \leq s$) is in $\{0, 1, 2, \dots, 9\}$, and $a_s > 0$. That is, we have proved the existence of a decimal representation for each natural n.

Uniqueness Assume that n has two decimal representations:

$$n = a_s 10^s + a_{s-1} 10^{s-1} + \cdots + a_1 10 + a_0,$$
$$n = b_u 10^u + b_{u-1} 10^{u-1} + \cdots + b_1 10 + b_0,$$

with $a_s > 0$ and $b_u > 0$. First note that

$$n = a_s 10^s + a_{s-1} 10^{s-1} + \cdots + a_1 10 + a_0 \geq 10^s$$

since $a_i \geq 0$ for $0 \leq i \leq s-1$ and $a_s \geq 1$. Using that all $a_i \leq 9$ and a geometric sum, we get

$$\begin{aligned} n &= a_s 10^s + a_{s-1} 10^{s-1} + \cdots + a_1 10 + a_0 \\ &\leq 9 \times 10^s + 9 \times 10^{s-1} + \cdots + 9 \times 10 + 9 \\ &= 9\left(\frac{10^{s+1} - 1}{10 - 1}\right) = 10^{s+1} - 1 < 10^{s+1}. \end{aligned}$$

Thus,

$$10^s \leq n < 10^{s+1}.$$

The exact same arguments show that

$$10^u \leq n < 10^{u+1}.$$

This implies that $u = s$ (why?). Therefore,

$$a_s 10^s + a_{s-1} 10^{s-1} + \cdots + a_1 10 + a_0 = b_s 10^s + b_{s-1} 10^{s-1} + \cdots + b_1 10 + b_0.$$

In particular,

$$a_0 - b_0 = b_s 10^s + b_{s-1} 10^{s-1} + \cdots + b_1 10 - \big(a_s 10^s + a_{s-1} 10^{s-1} + \cdots + a_1 10\big).$$

On the left-hand side we have a number whose absolute value is less than 9 (since a_0 and b_0 are both positive and less than 9). On the right-hand side we have a multiple of 10. The equality may hold only if both sides are 0. Thus, $a_0 = b_0$. We now have

$$a_s 10^s + a_{s-1} 10^{s-1} + \cdots + a_1 10 = b_s 10^s + b_{s-1} 10^{s-1} + \cdots + b_1 10.$$

Dividing both sides by 10 gives

$$a_s 10^{s-1} + a_{s-2} 10^{s-2} + \cdots + a_1 = b_s 10^{s-1} + b_{s-2} 10^{s-2} + \cdots + b_1.$$

We are now back to the initial situation with a_1 and b_1 playing the roles of a_0 and b_0, respectively. Using the same argument as above, we get $a_1 = b_1$. Repeating the procedure $s + 1$ times yields $a_i = b_i$ for $0 \leq i \leq s$. The decimal representation is unique. That concludes our proof.

Example 8.1 The proof of existence of a decimal representation is interesting in that it provides a method to find representations of a number. Moreover, the method may be applied to finding a representation in any base. The representation of n in base r (where r is any natural number) is given by

$$n = a_0 + a_1 r + \cdots + a_s r^s,$$

where a_i for $0 \leq i \leq s$ is in $\{0, 1, \ldots, r-1\}$, and $a_s \geq 1$.

We illustrate this point by applying the method to find a representation in base 3 for the number whose decimal representation is 121. First note that

$$3^4 \leq 121 < 3^5.$$

That is, $s = 4$, and the number of digits in base 3 is 5 ($s + 1$ in the notation above). We perform the division of 121 by 3^4 to get

$$121 = 3^4 + 40.$$

Thus, we get $a_4 = 1$. We now divide 40 by 3^3:

$$40 = 3^3 + 13.$$

That is, $a_3 = 1$. Then

$$13 = 3^2 + 4,$$

and $a_2 = 1$. Similarly, $a_1 = 1$ and $a_0 = 1$. That is, the base 3 representation of 121 is 11111.

We now turn to the decimal representation of real numbers. We start by defining

Integral part of a real
For any real number x, there exists an integer denoted by $[x]$ such that

$$[x] \leq x < [x] + 1.$$

In particular, $[x]$ is the largest integer smaller than x. We call $[x]$ the integral part of x.

Assume first that $x \geq 0$. If $x < 1$, then let $[x] = 0$. We have

$$0 \leq x < 0 + 1$$

for x in $[0, 1)$. Thus, for x in [0,1), $[x] = 0$. We now turn to $x \geq 1$. Define

$$B = \{n \geq 1 : n > x\}.$$

By the Archimedean property, B is nonempty. Since it is a subset of natural numbers, it has a minimum n_0. From $x \geq 1$ we have $n_0 > 1$. This implies that $n_0 - 1 \geq 1$. But $n_0 - 1$ is not in B. Thus, $n_0 - 1 \leq x < n_0$. Set $[x] = n_0 - 1$. With this definition, $[x]$ is an integer and is such that

$$[x] \leq x < [x] + 1.$$

For $x < 0$, we have $-x > 0$, and the existence of an integral part for x is an easy consequence of the existence of an integral part for $-x$. This is left as an exercise.

Note that for any real x, we have

$$x = [x] + \big(x - [x]\big).$$

The integral part may be represented using the decimal representation for naturals. We now need to represent $(x - [x])$ which is in $[0, 1)$. Putting together the decimal representation for $[x]$ and the decimal representation for $x - [x]$, we get a decimal representation for x.

Decimal representation of a real number
Let x be a real number in $[0, 1)$. Let the sequences a_i and d_i be defined by

$$\begin{aligned} a_i &= \left[10^i x\right] \quad \text{for } i \geq 1, \\ d_1 &= a_1, \\ d_i &= a_i - 10a_{i-1} \quad \text{for all } i \geq 2. \end{aligned}$$

Then the d_i are all in $\{0, 1, \dots, 9\}$, and $d_i < 9$ for infinitely many i. The sequence d_i gives the decimal representation

$$x = \sum_{i=1}^{\infty} \frac{d_i}{10^i}.$$

Moreover, this decimal representation is unique.

Fix x in $[0, 1)$. We will need five steps to show that x has a unique decimal representation.

Step 1. We show that all d_i are in $\{0, 1, \dots, 9\}$.

First note that the a_i and therefore the d_i are integers. By the definition of the integral part, we have

$$a_i \leq 10^i x < a_i + 1 \quad \text{for all } i \geq 1.$$

Our first step is to show that all d_i are in $\{0, 1, \dots, 9\}$. First note that since $x < 1$, we have

$$0 \leq d_1 = a_1 = [10x] \leq 10x < 10.$$

Thus, d_1 is an integer in $\{0, 1, \dots, 9\}$. We now deal with d_i for $i \geq 2$. We use

$$a_i \leq 10^i x < a_i + 1$$

and

$$a_{i-1} \leq 10^{i-1} x < a_{i-1} + 1.$$

First note that

$$10^i x - 10 = 10\big(10^{i-1} x - 1\big) < 10a_{i-1}.$$

Thus,

$$d_i = a_i - 10a_{i-1} < 10^i x - \big(10^i x - 10\big) = 10.$$

Since d_i is an integer strictly less than 10, it must be less than or equal to 9. We now show that it is also positive. We have

$$d_i = a_i - 10a_{i-1} > 10^i x - 1 - 10\big(10^{i-1} x\big) = -1.$$

Since d_i is an integer strictly larger than -1, it must be larger than or equal to 0. We have achieved our first goal: all d_i are in $\{0, 1, \dots, 9\}$.

Step 2. We prove that

$$a_i = d_i + 10d_{i-1} + \cdots + 10^{i-1} d_1 \quad \text{for all } i \geq 1. \tag{8.1}$$

By definition we have

$$d_1 = a_1,$$

and (8.1) holds for $i = 1$. Assume that (8.1) holds for i. Then we have

$$d_{i+1} = a_{i+1} - 10a_i = a_{i+1} - 10\big(d_i + 10d_{i-1} + \cdots + 10^{i-1} d_1\big)$$

by the induction hypothesis. Thus,

$$a_{i+1} = d_{i+1} + 10\big(d_i + 10d_{i-1} + \cdots + 10^{i-1} d_1\big) = d_{i+1} + 10d_i + \cdots + 10^i d_1,$$

and (8.1) holds for $i + 1$. The formula is proved by induction.

Step 3. We prove that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{10^i}.$$

Define the sequence s_i by

$$s_i = 10^{-i} a_i \quad \text{for } i \geq 1.$$

Note that

$$s_i = 10^{-i} a_i \leq 10^{-i} 10^i x = x$$

and that

$$s_i = 10^{-i} a_i > 10^{-i}\big(10^i x - 1\big) = x - 10^{-i}.$$

Thus,

$$x - 10^{-i} < s_i \leq x \quad \text{for all } i \geq 1. \tag{8.2}$$

By the squeezing principle the sequence s_i converges to x:

$$\lim_{i \to \infty} s_i = x.$$

Note that

$$s_i = 10^{-i} a_i = 10^{-i}\big(d_i + 10d_{i-1} + \cdots + 10^{i-1} d_1\big) = \sum_{j=1}^{i} \frac{d_j}{10^j}.$$

Letting i go to infinity, we get

$$x = \sum_{j=1}^{\infty} \frac{d_j}{10^j}.$$

That is, every real number in [0, 1) has a decimal expansion.

Step 4. We show that infinitely many of the d_i must be strictly less than 9.

By contradiction, assume that only finitely many d_i are strictly less than 9. Then there exists a natural n such that for all $i \geq n$, we have $d_i = 9$. Therefore,

$$x - s_n = \sum_{j=n+1}^{\infty} \frac{d_j}{10^j} = \sum_{j=n+1}^{\infty} \frac{9}{10^j} = \frac{9}{10^{n+1}} \frac{1}{1 - 1/10} = \frac{1}{10^n}.$$

On the other hand, by (8.2),

$$0 \leq x - s_n < 10^{-n}.$$

That is, $x - s_n = \frac{1}{10^n}$ and $x - s_n < 10^{-n}$. We reach a contradiction. Therefore, infinitely many of the d_i must be strictly less than 9.

Step 5. The decimal expansion is unique.

We have just shown that for any x in [0, 1), there are integers d_i in $\{0, 1 \ldots, 9\}$ such that infinitely many of the d_i are not 9 and such that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{10^i}.$$

Moreover, the sequence d_i may be computed using

$$a_i = \left[10^i x\right] \quad \text{for } i \geq 1$$

and

$$\begin{aligned} d_1 &= a_1, \\ d_i &= a_i - 10a_{i-1} \quad \text{for all } i \geq 2. \end{aligned}$$

Assume now that there is another decimal representation for x:

$$x = \sum_{i=1}^{\infty} \frac{d_i'}{10^i},$$

where the d_i' are integers in $\{0, 1 \ldots, 9\}$ such that infinitely many of the d_i' are not 9. We are now going to prove that for each $i \geq 1$, we have $d_i' = d_i$.

For any natural n, we have

$$10^n x = \sum_{i=1}^{\infty} 10^{n-i} d_i' = \sum_{i=1}^{n} 10^{n-i} d_i' + \sum_{i=n+1}^{\infty} 10^{n-i} d_i'.$$

Since there are infinitely many d_i' that are not 9, there are infinitely many i such that $i > n$ and $d_i' < 9$. Thus,

$$\sum_{i=n+1}^{\infty} 10^{n-i} d_i' < \sum_{i=n+1}^{\infty} 10^{n-i} \times 9 = 1.$$

Therefore,

$$\sum_{i=1}^{n} 10^{n-i} d_i' \leq 10^n x < \sum_{i=1}^{n} 10^{n-i} d_i' + 1.$$

Note also that

$$k = \sum_{i=1}^{n} 10^{n-i} d_i' = 10^{n-1} d_1' + 10^{n-2} d_2' + \cdots + d_n'$$

is an integer. That is, we have

$$k \leq 10^n x < k + 1,$$

where k is an integer. By the definition of the integral part of a real number, we have

$$\left[10^n x\right] = k = \sum_{i=1}^{n} 10^{n-i} d_i'. \tag{8.3}$$

We now are ready to prove by induction that for all $i \geq 1$, we have $d_i = d_i'$. Let $n = 1$ in (8.3) to get

$$[10x] = d_1'.$$

Thus, $d_1' = d_1$. Assume now that for $i \leq j$, $d_i = d_i'$. Let $n = j + 1$ in (8.3):

$$\begin{aligned}
\left[10^{j+1} x\right] &= \sum_{i=1}^{j+1} 10^{j+1-i} d_i' = \sum_{i=1}^{j} 10^{j+1-i} d_i + 10^{j+1-j-1} d_{j+1}' \\
&= 10 \sum_{i=1}^{j} 10^{j-i} d_i + d_{j+1}'.
\end{aligned}$$

However, (8.1) states that

$$a_j = d_j + 10 d_{j-1} + \cdots + 10^{j-1} d_1 = \sum_{i=1}^{j} 10^{j-i} d_i \quad \text{for all } j \geq 1.$$

Therefore,

$$\left[10^{j+1} x\right] = 10 a_j + d_{j+1}'.$$

That is,

$$d_{j+1}' = a_{j+1} - 10 a_j.$$

But d_{j+1} is, by definition, $a_{j+1} - 10 a_j$. Hence,

$$d_{j+1}' = d_{j+1}.$$

This concludes the proof that decimal expansions are unique.

Remark Putting together the decimal representation of the naturals and the decimal representation of real numbers in [0,1), we see that all positive real numbers can be represented by

$$b_0 + b_1 10 + b_2 10^2 + \cdots + b_s 10^s + \sum_{i=1}^{\infty} \frac{d_i}{10^i},$$

where $s \geq 0$ is an integer, the b_i and d_i are integers in $\{0, 1, 2, \ldots, 9\}$, and infinitely many of the d_i are not 9. It is a remarkable fact that ALL real numbers can be written in this simple form.

Example 8.2 Find d_i for $i = 1, 2, \ldots, 5$ for $x = \frac{1}{\sqrt{2}}$.

We first compute $a_1 = [10x]$. Note that

$$(10x)^2 = 50.$$

Thus,

$$7 < 10x < 8,$$

and $a_1 = 7$. This is also the first decimal $d_1 = 7$. We now compute $a_2 = [10^2 x]$. Using $(10^2 x)^2 = 5000$, we get that

$$70 < 10^2 x < 71.$$

So $a_2 = 70$ and $d_2 = a_2 - 10a_1 = 0$.

We have that $(10^3 x)^2 = 500000$ and

$$707 < 10^3 x < 708.$$

So $a_3 = 707$ and $d_3 = a_3 - 10a_2 = 707 - 700 = 7$. And so on. We get

$$x = \frac{\sqrt{2}}{2} = 0.70710\ldots.$$

Application 8.1 The elementary way to obtain decimal expansions for rational numbers uses repeated Euclidean divisions (Note that this method cannot be defined for irrationals!). For instance, by dividing 1 by 3 one gets $1/3 = 0.3333\ldots$. We are now going to check that this algorithm gives the same result as the general method described above.

Assume that the rational number x in $(0, 1)$ can be represented by a/b where $a < b$ are natural numbers. We start by performing the division of $10a$ by b to get

$$10a = q_1 b + r_1,$$

where $0 \leq r_1 < b$. We then divide $10r_1$ by b to get

$$10r_1 = q_2 b + r_2,$$

where $0 \leq r_2 < b$. We define inductively the sequences q_i and r_i of positive or zero integers by performing successive divisions. More precisely, let $r_0 = a$, and for $i \geq 1$, let

$$10r_{i-1} = q_i b + r_i.$$

Note that for all $i \geq 0$, we have $0 \leq r_i < b$. Observe also that for all $i \geq 1$, we must have $q_i < 10$. By contradiction, assume that $q_i \geq 10$ for some $i \geq 1$. Then

$$10r_{i-1} = q_i b + r_i \geq 10b$$

and $r_{i-1} \geq b$. This is a contradiction. Thus, $0 \leq q_i < 10$ for all $i \geq 1$.

We now prove by induction that

$$x = \sum_{i=1}^{n} \frac{q_i}{10^i} + \frac{r_n}{10^n b} \quad \text{for all } n \geq 1. \tag{8.4}$$

We have

$$10r_0 = 10a = q_1 b + r_1.$$

Dividing by $10b$, we get

$$x = \frac{a}{b} = \frac{q_1}{10} + \frac{r_1}{10b}.$$

Therefore, (8.4) holds for $n = 1$.

Assume now that (8.4) holds for n. On the other hand, by definition

$$10r_n = q_{n+1} b + r_{n+1}.$$

Thus,

$$r_n = \frac{q_{n+1}}{10} b + \frac{r_{n+1}}{10}.$$

The induction hypothesis is

$$x = \sum_{i=1}^{n} \frac{q_i}{10^i} + \frac{r_n}{10^n b}.$$

We now express r_n in function of r_{n+1} to get

$$x = \sum_{i=1}^{n} \frac{q_i}{10^i} + \frac{1}{10^n b} \left(\frac{q_{n+1}}{10} b + \frac{r_{n+1}}{10} \right).$$

That is,

$$x = \sum_{i=1}^{n+1} \frac{q_i}{10^i} + \frac{r_{n+1}}{10^{n+1} b}.$$

We have proved (8.4) by induction.

For $n \geq 1$, let

$$t_n = \sum_{i=1}^{n} \frac{q_i}{10^i}.$$

Using that $r_n < b$ in (8.4), we get

$$0 \leq x - t_n < \frac{1}{10^n}. \tag{8.5}$$

In particular,

$$\lim_{n\to\infty} t_n = x.$$

That is,

$$x = \sum_{i=1}^{\infty} \frac{q_i}{10^i}.$$

The exactly same argument used above for the d_i shows that infinitely many of the q_i are not 9 (here (8.5) and t_n play the roles of (8.2) and s_n there). Therefore, we have found a decimal representation

$$x = \sum_{i=1}^{\infty} \frac{q_i}{10^i},$$

where the q_i are integers in $\{0, 1, \ldots, 9\}$, and infinitely many of them are not 9. But we know that such a representation is unique. Therefore, $q_i = d_i$ for all $i \geq 1$. This proves that the elementary algorithm used to get decimal expansions for rationals is legitimate.

Application 8.2 Are the decimal representations of rationals special in some regard? The answer is of course yes. A rational has a *periodic* decimal expansion. That is, if

$$x = \sum_{i \geq 1} \frac{d_i}{10^i}$$

is rational, then there are natural numbers p and k such that

$$d_{i+p} = d_i \quad \text{for all } i \geq k.$$

For instance, $1/6 = 0.16666\ldots$. We have that $k = 2$ (the periodicity starts at the second decimal) and $p = 1$ (the period is 1). The periodicity property of the rationals is a direct consequence of the division algorithm seen in Application 8.1. In order to find the sequence of decimals for the rational a/b, we divide by b the successive remainders. But each remainder must be strictly less than b. Thus, after at most b divisions we will get a remainder that we already got before. From then on, the sequence of decimals is going to repeat itself.

Application 8.3 Are the rationals the only numbers with periodic expansions? The answer is yes.

Assume that x has a periodic expansion. That is, there are natural numbers p and k such that

$$d_{i+p} = d_i \quad \text{for all } i \geq k.$$

There are two possibilities. Either $k = 1$, i.e., the periodicity starts at the first decimal, or $k > 1$. The proof for both cases is essentially the same, so we will write the proof for $k = 1$ only. We have

$$x = \sum_{i \geq 1} \frac{d_i}{10^i} = \sum_{n=0}^{\infty} \sum_{i=np+1}^{(n+1)p} \frac{d_i}{10^i}.$$

That is, we are summing the series by packets of p elements. This is correct when the series converges absolutely (why is this the case here?) but might give a wrong result in other cases (see the exercises). Since the d_i are assumed to be periodic, we get

$$\sum_{i=np+1}^{(n+1)p} \frac{d_i}{10^i} = \frac{d_1}{10^{np+1}} + \frac{d_2}{10^{np+2}} + \cdots + \frac{d_p}{10^{(n+1)p}}$$
$$= \frac{1}{10^{np}} \left(\frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_p}{10^p} \right).$$

Let

$$c = \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_p}{10^p}.$$

Summing a geometric series, we get

$$x = \sum_{n=0}^{\infty} \sum_{i=np+1}^{(n+1)p} \frac{d_i}{10^i} = \sum_{n=0}^{\infty} \frac{c}{10^{np}} = \frac{c}{1 - \frac{1}{10^p}}.$$

Since c is rational and so is $1 - \frac{1}{10^p}$, x must be a rational as well, and we are done.

Exercises

1. (a) Find the base 4 representation of 121.
   (b) What is the decimal representation of the number whose representation in base 2 is 1010101?
2. Assume that for a natural number n, there are two positive integers s and u such that

   $$10^s \leq n < 10^{s+1}$$

   and

   $$10^u \leq n < 10^{u+1}.$$

   Show that $s = u$.
3. (a) Let $x < 0$. Show that $[x]$ exists. In particular, express $[x]$ in function of $[-x]$.
   (b) Sketch the graph of the function 'integral part'.
4. Show that the integral part of a real is unique.
5. Show that for every natural n, we have

   $$10^n > n.$$

6. Let x be in $[0, 1)$ and assume that its decimal representation terminates. That is, there is a natural n such that

$$x = \sum_{i=1}^{n} \frac{d_i}{10^i}.$$

Show that there is a natural number k such that

$$x = \frac{k}{10^n}.$$

7. Let x have the decimal representation

$$x = 0.123456789123456789\ldots,$$

where 123456789 are repeated in the same order indefinitely. Express s as a fraction.

8. Let x have the decimal representation

$$x = 0.101001000100001\ldots,$$

where the number of 0's between two 1's increases by 1 at every stage. Show that x is irrational.

9. Find the decimal representation of $\frac{1}{\sqrt{3}}$.

10. (a) Find the binary representation of $2/3$. That is, find the sequence d_i such that

$$2/3 = \sum_{i=0} \frac{d_i}{2^i},$$

where the d_i are in $\{0, 1\}$, and infinitely many of them are 0's.

(b) The binary representation of x is

$$x = 0.1001001001\ldots.$$

Find the decimal representation of x.

11. Assume that

$$x = \sum_{i=1}^{\infty} \frac{c_i}{10^i},$$

where only finitely many c_i are not 9. Find the decimal representation for x.

12. Consider the series $\sum_{k=0}^{\infty}(-1)^k$.
(a) Sum the series in packets so that the sum of packets converges.
(b) Is the sum of the packets the same as the sum of the series in this case?

Chapter 9
Countable and Uncountable Sets

We have already come across several infinite sets: $\mathbf{N}$ the set of naturals, $\mathbf{Z}$ the set of integers, $\mathbf{Q}$ the set of rationals, and $\mathbf{R}$ the set of reals. It turns out that, in some sense to be made precise, the first three sets may be said to have the same cardinality, while the last one has a strictly larger cardinality. We start with a definition.

One-to-one and onto functions
Let A and B be two sets, and let f be a function $f : A \to B$. The function f is said to be one-to-one if $f(x) = f(y)$ implies $x = y$. The function f is said to be onto if for every b in B, there is a in A such that $f(a) = b$.

Example 9.1 Let $A = \mathbf{N}$, and B be the odd naturals. Define $f : A \to B$ by

$$f(n) = 2n + 1.$$

Show that f is one-to-one and onto.

Assume that for two naturals x and y, we have $f(x) = f(y)$. Then

$$2x + 1 = 2y + 1,$$

and therefore $x = y$. Thus, f is one-to-one.

Let b be in B. By definition b is odd. Thus, there is a positive integer a such that

$$b = 2a + 1.$$

So $b = f(a)$. This proves that f is onto as well.

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Inverse functions
Let A and B be two sets, and let f be a function $f : A \to B$. If f is one-to-one and onto, then f is said to be a bijection. For such an f, there exists an inverse function denoted by f^{-1} such that $f^{-1} : B \to A$ and such that

$$f\big(f^{-1}(b)\big) = b \quad \text{for every } b \text{ in } B$$

and

$$f^{-1}\big(f(a)\big) = a \quad \text{for every } a \text{ in } A.$$

Assume that f is a bijection. Then for every b in B, there is a in A such that $f(a) = b$. This is so because f is onto. Moreover, a is unique: if $f(a) = f(a_1) = b$, then $a = a_1$ since f is one-to-one. Therefore, we may define

$$f^{-1}(b) = a \tag{9.1}$$

for every b in B. If we plug $b = f(a)$ into (9.1), we get

$$f^{-1}\big(f(a)\big) = a.$$

On the other hand, if we take the images by f of both sides of (9.1), we get

$$f\big(f^{-1}\big)(b) = f(a) = b$$

for every b in B. This completes our proof.

We now introduce the notion of cardinality.

Cardinality
Two sets A and B are said to have the same cardinality if there is a function $f : A \to B$ which is one-to-one and onto.

We now state and prove several consequences of our definition for cardinality.

C1. Any set A has the same cardinality as itself.

Define $f : A \to A$ as $f(a) = a$ for every a in A. It is easy to check that f is one-to-one and onto. Thus, A has the same cardinality as A.

C2. If A has the same cardinality as B, then B has the same cardinality as A.

If A has the same cardinality as B, then there is a function $f : A \to B$ which is one-to-one and onto. Then the inverse function f^{-1} of f is well defined and is also one-to-one and onto. Moreover, $f^{-1} : B \to A$. Thus, B has the same cardinality as A.

C3. Show that if A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C.

Assuming that A has the same cardinality as B and B has the same cardinality as C, there are bijections f and g such that $f : A \to B$ and $g : B \to C$. For every a in A, $f(a)$ is in B, and we may define

$$h(a) = g\big(f(a)\big).$$

Note that $h : A \to C$. It is easy to check that h is one-to-one and onto, and this is left as an exercise. This shows that A has the same cardinality as C, and we are done.

Follows a mathematical definition of finiteness.

Finite sets
A set A is said to be finite if it is the empty set or if there is a natural n such that A has same cardinality as $\{1, 2, \ldots, n\}$. A set which is not finite will be said to be infinite.

A finite set A is such that there is a function $f : \{1, 2, \ldots, n\} \to A$ which is one-to-one and onto. In particular, for every a in A, there is a unique (why?) i in $\{1, 2, \ldots, n\}$ such that $f(i) = a$. In other words, each element of A has a unique label i.

Example 9.2 Show that $\{1, 2, \ldots, n\}$ has the same cardinality as $\{1, 2, \ldots, k\}$ if and only if $n = k$.

If $n = k$, then it is easy to see that $\{1, 2, \ldots, n\}$ has the same cardinality as $\{1, 2, \ldots, k\}$: define f by $f(i) = i$ for every i in $\{1, 2, \ldots, n\}$. This is a one-to-one and an onto function.

For the converse, assume that $\{1, 2, \ldots, n\}$ has the same cardinality as $\{1, 2, \ldots, k\}$. By definition there is a function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ which one-to-one and onto. Since f is one-to-one, all the $f(i)$ must be distinct. Otherwise, we would have $f(i) = f(j)$ for $i \neq j$, contradicting the one-to-one property. Thus, we must have $k \geq n$ since we have n distinct $f(i)$ in $\{1, 2, \ldots, k\}$. On the other hand, if $k > n$, then there is at least one element of $\{1, 2, \ldots, k\}$ which is not the image of an element in $\{1, 2, \ldots, n\}$. This is so because the set $\{f(1), f(2), \ldots, f(n)\}$ has exactly $n < k$ elements. Then f cannot be onto. This is a contradiction. Therefore, $k \leq n$. This, together with $k \geq n$, implies that $k = n$, and we are done.

Countable sets
A set A is said to be countable if it has the same cardinality as the set of naturals $\mathbf{N}$.

Example 9.3 In Example 9.1 we have shown that the set of odd integers has the same cardinality as the set of the naturals. Thus, the set of odd integers is countable. Note that the set of odd numbers is strictly included in the set of naturals, but according to our definition, they have the same cardinality! No such monkey business may happen for finite sets as will be shown in the exercises.

Example 9.4 We will now show that the set of integers $\mathbf{Z}$ is countable and therefore has the same cardinality as $\mathbf{N}$.

Let $f : \mathbf{Z} \to \mathbf{N}$ be defined as follows. If $n < 0$, then $f(n) = -2n$. If $n \geq 0$, then $f(n) = 2n + 1$. We now show that f is one-to-one. Assume that $f(a) = f(b)$. We cannot have $-2a = 2b + 1$ or $2a + 1 = -2b$ (why not?). Thus, either $f(a) = -2a$ and $f(b) = -2b$, so that $a = b$, or $f(a) = 2a + 1$ and $f(b) = 2b + 1$, so that $a = b$. Thus, f is one-to-one.

We now show that f is onto. Take a natural n. Then if n is odd, there is a natural a such that $n = 2a + 1$ and $f(a) = n$. If n is even or 0, then set $a = -n/2$ and $f(a) = -2a = n$. Therefore, f is also onto.

We will now show that the situation in Example 9.3 is actually the rule: any infinite subset of $\mathbf{N}$ has the same cardinality as $\mathbf{N}$.

Infinite subsets of naturals
Let S be an infinite subset of $\mathbf{N}$. Then S has the same cardinality as $\mathbf{N}$. That is, S is countable.

This says that any infinite subset of $\mathbf{N}$ has the same cardinality as $\mathbf{N}$. That is, the cardinality of $\mathbf{N}$ is the 'smallest' infinite.

We now prove this important fact. Since S is infinite it is nonempty and since it is a subset of the naturals, it has a minimum (by the well-ordering principle) that we denote by k_1. Note that $k_1 \geq 1$. Take k_1 out of the set S to get a new set denoted by $S - \{k_1\}$. This new set is not empty (why?) and therefore has a minimum k_2. Since k_1 is not in $S - \{k_1\}$, we must have $k_2 > k_1$. Thus, $k_2 \geq 2$. We define like this an infinite sequence k_n such that k_n is the minimum of the set $S - \{k_1, k_2, \ldots, k_{n-1}\}$. For every n, the set $S - \{k_1, k_2, \ldots, k_{n-1}\}$ cannot be empty (since S is infinite), and therefore it has a minimum. Note that by construction we have $k_1 < k_2 < \cdots$, and in particular all k_i are distinct. Moreover, for every i, $k_i \geq i$. The proof of the last statement is left as an exercise. Let K be the set

$$K = \{k_1, k_2, \ldots, k_n, \ldots\}.$$

We now show by contradiction that K is S. Assume that K is strictly included in S. Then there is z in S which is not in K. As noted above, we have $k_z \geq z$, but since z is not in K, we must have $k_z > z$. By construction k_z is the minimum of $S - \{k_1, k_2, \ldots, k_{z-1}\}$. Since $k_z > z$, z cannot be in the set $S - \{k_1, k_2, \ldots, k_{z-1}\}$. But since z is in S, it must be in the set $\{k_1, k_2, \ldots, k_{z-1}\}$, that is, in K. Thus, we reach a contradiction. All elements of S are in K.

Define now $f : \mathbf{N} \to S$ by $f(i) = k_i$. All the k_i are distinct, so f is one-to-one. Since $S = K$, f is also onto. Therefore, S has the same cardinality as $\mathbf{N}$.

Application 9.1 Any infinite subset of a countable set is countable.

This is in fact an easy consequence of the result above. Let S be a countable set, and let A be an infinite subset of S. There is a function $f : \mathbf{N} \to S$ which is one-to-one and onto. Let T be the subset of naturals such that

$$T = \left\{n \in \mathbf{N} : f(n) \in A\right\}.$$

The set T must be infinite. Assume by contradiction that T is finite. Then there is a natural k and naturals $n_1, n_2, \ldots, n_k$ such that $T = \{n_1, n_2 \ldots, n_k\}$. Therefore, $A = \{f(n_1), f(n_2) \ldots, f(n_k)\}$, and A is a finite set. This provides a contradiction since A is infinite.

Since T is an infinite subset of the naturals, T must be countable. Let g be the function f restricted to T. That is, let $g : T \to A$ be such that for all n in T, we have $g(n) = f(n)$. Note that g is one-to-one (since f is one-to-one). Let a be in A. f is onto, so there is n in $\mathbf{N}$ such that $f(n) = a$. Since $f(n)$ is in A, n is in T. Therefore, $g(n) = f(n) = a$. That is, g is onto. So A has the same cardinality as T, which has the same cardinality as $\mathbf{N}$. By C3, A has the same cardinality as $\mathbf{N}$: A is countable, and we are done.

We next state three convenient criteria for countability.

Lemma *Assume that S is an infinite set. Then the three following properties are equivalent.*

(i) *S is countable.*
(ii) *There exists a subset T of $\mathbf{N}$ and a function $f : T \to S$ that is onto.*
(iii) *There exists a function $g : S \to \mathbf{N}$ which is one-to-one.*

We now prove the lemma. Assume (i). Then there exists a function $f : \mathbf{N} \to S$ which is one-to-one and onto. Therefore, (ii) holds with $T = \mathbf{N}$.

Assume now that (ii) holds. Let s be an element in S, and let A_s be

$$A_s = \left\{n \in T : f(n) = s\right\}.$$

Since f is onto, A_s is a nonempty subset of $\mathbf{N}$, and it has a minimum that we denote by $g(s)$. This is true for any s in S; therefore, this defines a function $g : S \to \mathbf{N}$. We now show that g is one-to-one. Assume that $g(a) = g(b)$. By definition $g(a)$ is the minimum of the set A_a, and therefore, $f(g(a)) = a$. Similarly, $f(g(b))$ must be b. But if $g(a) = g(b)$, we must have $f(g(a)) = f(g(b))$, and therefore, $a = b$. Thus, (iii) holds.

Finally, assume (iii). Define the set U by

$$U = \left\{g(s), s \in S\right\}.$$

Note that $g : S \to U$ is onto. By hypothesis it is also one-to-one. Therefore, S has the same cardinality as U. Since S is assumed to be infinite, so must be U (since g is one-to-one). Therefore, U is an infinite subset of $\mathbf{N}$ and so is countable. Since S has the same cardinality as U, which has the same cardinality as $\mathbf{N}$, S has the same cardinality as $\mathbf{N}$. Thus, (i) holds, and the lemma is proved.

The lemma is helpful in proving the following.

Union of countable sets
A countable union or a finite union of countable sets is countable.

We will prove the result for a countable union. The proof for a finite union can be easily adapted. What we mean by countable union is that the index set of the union is countable. The general case is

$$A = \bigcup_{i \in \mathbf{N}} A_i,$$

so that the index set is $\mathbf{N}$. Assume that each A_i is countable. For each i in $\mathbf{N}$, there exists a function $f_i : \mathbf{N} \to A_i$ which is one-to-one and onto. Consider the following subset of $\mathbf{N}$:

$$T = \left\{2^i 3^j : i, j \in \mathbf{N}\right\}.$$

That is, T is the set of naturals whose unique decomposition in prime numbers only has the primes 2 and 3. The fact that each natural number has a unique factorization in prime numbers is known as the fundamental theorem of arithmetic (for a proof, see, for instance, 'An introduction to Number Theory' by Hardy and Wright (1980), Oxford University Press).

Define the function $f : T \to A$ by

$$f\left(2^i 3^j\right) = f_i(j).$$

Take now a an element in A. Then there is i in $\mathbf{N}$ such that a belongs to A_i, and there is j in $\mathbf{N}$ such that $f_i(j) = a$ since we assume that f_i is onto. Therefore, $a = f(2^i 3^j)$, and f is onto. Thus, by criterion (ii) of the lemma, A is countable.

Cartesian product
If A and B are countable, then the Cartesian product $A \times B$ defined by

$$A \times B = \left\{(a, b) : a \in A, b \in B\right\}$$

is also countable.

This is again a consequence of the lemma. Since A and B are countable, there are functions f_1 and f_2 such that $f_1 : \mathbf{N} \to A$ and $f_2 : \mathbf{N} \to B$ are both one-to-one and onto. Define $f : \mathbf{N} \times \mathbf{N} \to A \times B$ by $f(i, j) = (f_1(i), f_2(j))$. It is left as an exercise to show that f is one-to-one and onto. Thus, $A \times B$ is countable if and only if $\mathbf{N} \times \mathbf{N}$ is countable. We now prove that $\mathbf{N} \times \mathbf{N}$ is countable. Define $g : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ by $g(i, j) = 2^i 3^j$. Then by the fundamental theorem of arithmetic, g is one-to-one. Thus, by the lemma (iii), $\mathbf{N} \times \mathbf{N}$ is countable, and so is $A \times B$.

Example 9.5 In Example 9.4 we have shown that $\mathbf{Z}$ is countable. Therefore, $\mathbf{Z} \times \mathbf{Z}$ is also countable.

The rationals
The set $\mathbf{Q}$ of rationals is countable.

Let $\mathbf{Z}^*$ be the set of nonzero integers. By Application 9.1 the set $\mathbf{Z} \times \mathbf{Z}^*$ is countable since it is an infinite subset of the countable set $\mathbf{Z} \times \mathbf{Z}$. Thus, there is a function $g : \mathbf{N} \to \mathbf{Z} \times \mathbf{Z}^*$ which is one-to-one and onto. Consider now $f : \mathbf{Z} \times \mathbf{Z}^* \to \mathbf{Q}$ defined by $f(a, b) = a/b$ and let $h : \mathbf{N} \to \mathbf{Q}$ be defined by $h(i) = f(g(i))$. By (ii) of the lemma, it is enough to show that h is onto. Let r be a rational. Then there is (a, b) in $\mathbf{Z} \times \mathbf{Z}^*$ such that $r = a/b$. That is, $r = f(a, b)$. Since g is onto, there is i in $\mathbf{N}$ such that $g(i) = (a, b)$. Thus, $r = f(g(i)) = h(i)$. This proves that h is onto. Thus, $\mathbf{Q}$ is countable.

Example 9.6 The proof above that the set of rationals is a countable set is rather abstract. We have proved that there is a function $h : \mathbf{N} \to \mathbf{Q}$ that is one-to-one and onto, but we have not said what h looks like. We are now going to be more explicit. We label the elements of $\mathbf{N} \times \mathbf{N}$ as follows:

$$\begin{array}{llll}
q_1 = (1, 1), & & & \\
q_2 = (1, 2), & q_3 = (2, 1), & & \\
q_4 = (1, 3), & q_5 = (2, 2), & q_6 = (3, 1), & \\
q_7 = (1, 4), & q_8 = (2, 3), & q_9 = (3, 2), & q_{10} = (4, 1),
\end{array}$$

and so on. This gives an explicit way of labeling all elements of $\mathbf{N} \times \mathbf{N}$ and therefore all positive rationals.

We are now going to see our first infinite set which is not countable.

Example 9.7 Let S be the set of infinite sequences of 0's and 1's. For instance, the sequence $s = (0, 1, 0, 1, 0, 1, \ldots)$ is in S. Maybe surprisingly, S is not countable: there are too many elements in S to have one-to-one correspondence with $\mathbf{N}$. We do a proof by contradiction. Assume that S is countable. Then there exists a function $f : \mathbf{N} \to S$ which is one-to-one and onto. Note that $f(1)$ is an infinite sequence of 0's and 1's, and we write $f(1) = (f_1(1), f_2(1), \ldots)$, where $f_i(1)$ is 0 or 1 for each natural i. More generally, for any i in $\mathbf{N}$, we write $f(i) = (f_1(i), f_2(i), f_3(i), \ldots)$. We now define a new sequence a in S by setting

$$a_1 = 1 - f_1(1), \qquad a_2 = 1 - f_2(2), \qquad a_3 = 1 - f_3(3), \ldots.$$

More generally, we set $a_i = 1 - f_i(i)$ for every natural i. Note that since $f_i(i)$ is 0 or 1, so is a_i for every i. That is, the sequence a is in S. Since f is onto, there is j in $\mathbf{N}$ such that $a = f(j)$. But $a_j = 1 - f_j(j) \neq f_j(j)$, so $a \neq f(j)$. We have a contradiction. This proves that S is not countable.

The reals
The set $\mathbf{R}$ of reals is not countable.

We use the result in Example 9.7 to prove this result. It is actually enough to show that the set of reals in [0, 1] is not countable (why?). Assume, by contradiction, that

[0, 1] is countable. Then there is a function $g : [0, 1] \to \mathbf{N}$ which is one-to-one and onto. Consider the function $f : S \to [0, 1]$, where S is the set of infinite sequences of 0's and 1's, and f is defined by

$$f(s) = \sum_{i=1}^{\infty} \frac{s_i}{10^i}$$

for every $s = (s_1, s_2, \ldots)$ in S. We now show that f is one-to-one. Assume that for a and b in S, we have $f(a) = f(b)$. Let

$$x = \sum_{i=1}^{\infty} \frac{a_i}{10^i} = \sum_{i=1}^{\infty} \frac{b_i}{10^i}.$$

That is, a and b are decimal representations of the real x. But we know that a decimal representation is unique (provided that infinitely many of the digits are not 9, but here we have only 0's and 1's). Thus, $a = b$, and f is one-to-one. Define now the function $h : S \to \mathbf{N}$ by $h(s) = g(f(s))$. Assume that $h(a) = h(b)$ for a and b in S. That is, $g(f(a)) = g(f(b))$. Since g is one-to-one, we have $f(a) = f(b)$, and since f is one-to-one, $a = b$. That is, h is one-to-one. By the lemma, S is countable, but we know that it is not by Example 9.7. Thus, we have a contradiction, and [0, 1] is uncountable. Therefore, $\mathbf{R}$ is also uncountable.

Example 9.8 The set of irrational reals is uncountable.

The set of reals is the union of the set of rationals and the set of irrationals. The set of rationals is countable. If the set of irrationals were countable, then the union of rationals and irrationals would be countable as well (since the union of two countable sets is countable). The reals would be countable. That is not true. Hence, the set of irrationals is uncountable.

A natural question is: are there sets that have even more elements than the reals? The answer is yes. Actually, it is always possible to find sets with more elements than a given set. Given a set X, we define the power set of X, $\mathcal{P}(X)$, as the set of all subsets of X. For instance, if $X = \{1, 2, 3\}$, then

$$\mathcal{P}(X) = \big\{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\big\}.$$

It is clear that $\mathcal{P}(X)$ has strictly more elements than X and cannot have the same cardinality as X. This is in fact always true.

Power sets
For any set X, its power set $\mathcal{P}(X)$ has a strictly larger cardinality than X in the sense that there exists a one-to-one function from X to $\mathcal{P}(X)$ but no onto function.

Note that if $h : A \to B$ is one-to-one and if

$$C = \big\{h(a) : a \in A\big\},$$

then $h : A \to C$ is one-to-one and onto. Thus, A has the same cardinality as C which is a subset of B. This is why it makes sense to say that the cardinality of A is less than the cardinality of B if there is a one-to-one function from A to B.

We now prove that a set has a strictly smaller cardinality than its power set. If $X = \emptyset$, then $\mathcal{P}(X) = \{X\}$. The cardinality of X is 0, and the cardinality of $\mathcal{P}(X)$ is 1.

If $X \neq \emptyset$, then define $f : X \to \mathcal{P}(X)$ by setting for every x in X $f(x) = \{x\}$. Clearly, f is one-to-one.

We now show that there is no function $g : X \to \mathcal{P}(X)$ which is onto. By contradiction assume that such a function exists. Define

$$Y = \big\{x \in X : x \notin g(x)\big\}.$$

Note that Y belongs to $\mathcal{P}(X)$, and since g is onto, there is a in X such that $g(a) = Y$. If a belongs to Y, we have that a does not belong to $g(a)$ but $g(a)$ is Y, so that is not possible. Thus, a does not belong to Y. That is, a belongs to $g(a)$ which is Y. This is not possible either. Thus, we have a contradiction. There is no $g : X \to \mathcal{P}(X)$ which is onto.

The result above shows that the cardinality of X is strictly less than its power set $\mathcal{P}(X)$, whose cardinality is strictly less than its own power set $\mathcal{P}(\mathcal{P}(X))$, and so on. Thus, it is always possible to find a set with a strictly larger cardinality.

Exercises

1. Show that the function $f : A \to B$ is one-to-one and onto if and only if for every b in B, there is a unique a in A such that $f(a) = b$.
2. Assume that f and g are bijections such that $f : A \to B$ and $g : B \to C$. For every a in A, define
$$h(a) = g\big(f(a)\big).$$
Show that h is also a bijection.
3. Let $A \subset B$ and assume that A is finite. Then A and B have the same cardinality if and only if $A = B$.
4. Let $f : \mathbf{N} \to S$ for some set S. Let
$$F = \big\{f(i), i \in \mathbf{N}\big\}.$$
Show that F is countable or finite.
5. Show that
$$B = \big\{5^i 7^j : i, j \in \mathbf{N}\big\}$$
is countable.
6. Show that if k_i is a strictly increasing sequence of naturals, then $k_i \geq i$ for every $i \geq 1$.
7. Let $f_1 : \mathbf{N} \to A$ and $f_2 : \mathbf{N} \to B$ be one-to-one and onto. Define $f : \mathbf{N} \times \mathbf{N} \to A \times B$ by $f(i, j) = (f_1(i), f_2(j))$. Show that f is one-to-one and onto.
8. Define $g : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ by $g(i, j) = 2^i 3^j$.
   (a) Show that g is one-to-one.

(b) Is g onto?

9. (a) Show that

$$\mathbf{N}^3 = \mathbf{N} \times \mathbf{N} \times \mathbf{N} = \left\{(a, b, c) : a \in \mathbf{N}, b \in \mathbf{N}, c \in \mathbf{N}\right\}$$

is countable.

(b) Generalize (a).

10. Assume that $g : A \to B$ and $f : B \to C$ are onto. Let $h : A \to C$ be defined by $h(a) = f(g(a))$. Prove that h is onto.

11. Assume that $A \subset B$ and that A is infinite and uncountable. Prove that B is uncountable.

12. Is $\mathcal{P}(\mathbf{N})$ countable?

13. Given a nonempty set X, let

$$\{0, 1\}^X = \left\{f : f : X \to \{0, 1\}\right\}$$

be the set of functions from X to $\{0, 1\}$. In this exercise we are going to show that $\mathcal{P}(X)$ has the same cardinality as $\{0, 1\}^X$.

(a) Given A a subset of X, let 1_A be the indicator function of A. It is defined by $1_A(x) = 0$ if x does not belong to A and $1_A(x) = 1$ if x belongs to A. Show that 1_A belongs to $\{0, 1\}^X$.

(b) Define the function $F : \mathcal{P}(X) \to \{0, 1\}^X$ by $F(A) = 1_A$ for each A in $\mathcal{P}(X)$. Show that F is one-to-one.

(c) Let f be in $\{0, 1\}^X$. Define

$$A = \left\{x \in X : f(x) = 1\right\}.$$

Show that $f = 1_A$.

(d) Conclude.

Further Reading

On number systems:

S. Feferman (1964), *The Number Systems*, Addison-Wesley.
J.H. Goodfriend (2005), *Gateway to Higher Mathematics*, Jones and Bartlett.

More advanced analysis books:

S. Krantz (1991), *Real Analysis and Foundations*, CRC Press.
W. Rudin (1976), *Principles of Mathematical Analysis*, Third edition, Mc Graw-Hill.

To learn a lot more about the mathematics around π:

P. Eymard and J.P. Lafon (2004), *The Number π*, American Mathematical Society.

A fascinating little book on infinite series and infinite products:

J.M. Hyslop (1945), *Infinite Series*, Intersciences publishers.

A great book on Euler's life and work:

W. Dunham (1999), *Euler: The Master of Us All*, Mathematical Association of America.

R.B. Schinazi, *From Calculus to Analysis*,
DOI 10.1007/978-0-8176-8289-7,

List of Mathematicians Cited in the Text

Abel, Niels Henrik (1802–1829)
Archimedes of Syracuse (c. 287 BC–c. 212 BC)
Bernoulli, James (1654–1705)
Bolzano, Bernard (1781–1848)
Cantor, Georg (1845–1918)
Cauchy, Augustin-Louis (1789–1857)
Darboux, Gaston (1842–1917)
Euclid (ca. 325–ca. 270 BC)
Euler, Leonard (1707–1783)
Lagrange, Joseph Louis (1736–1813)
Lipschitz, Rudolph (1831–1904)
Mersenne, Marin (1588–1648)
Peano, Giusepe (1858–1932)
Ramanujan, Srinivasa (1887–1920)
Riemann, Bernhard (1826–1866)
Rolle, Michel (1652–1719)
Stirling, James (1692–1770)
Viète, Francois (1540–1603)
Wallis, John (1616–1703)
Weierstrass, Karl (1815–1897)

R.B. Schinazi, *From Calculus to Analysis*,
DOI 10.1007/978-0-8176-8289-7,

Index

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